Albert-Ludwigs-Universität Freiburg
Institut für Informatik
Prof. Dr. Susanne Albers
Dr. Alexander Souza

SS 2008

Combinatorial Optimization

Exercise 1 (Degrees)

Let $G = (V, E)$ be an undirected graph on n vertices and m edges. Show that

(1) $\sum_{v \in V} \deg(v) = 2m$, and

(2) the number of vertices with odd degree is even.

Exercise 2 (Cycles in Graphs)

Given a directed graph G on n vertices and m edges, say, show that there is a linear-time algorithm, i.e., the running time is $O(n + m)$, that finds a cycle in G or decides that none exists.

Exercise 3 (Running Time)

Suppose we are given a graph $G = (V, E)$ on n vertices and m edges with a weight function $w : E \to \mathbb{N}$, where we assume that this function is given as a vector $w = (w_1, \ldots, w_m)$. We are further given an algorithm that solves a certain problem and takes running time proportional to $n^2 W$, where $W = \sum_{e \in E} w(e)$. Is this a polynomial time algorithm?

Exercise 4 (Job Assignment)

The job assignment problem from the introduction

$$\begin{aligned} \text{minimize} \quad & \max_i \sum_j x_{ij} \\ \text{subject to} \quad & \sum_{i \in S_j} x_{ij} = p_j \\ & x_{ij} \geq 0 \qquad i = 1, \ldots, m,\ j = 1, \ldots, n \end{aligned}$$

is almost formulated as an LP, except that "max" appears in the objective function. Can you formulate the problem as an LP?

Exercise 5 (Polyhedra)

Prove Lemma 2.2 from the lecture: Let $P = \{x : Ax \leq b\}$ be a polyhedron and $F \subseteq P$. Then the following statements are equivalent:

(1) F is a face of P.

(2) There is a vector c with $\delta := \max\{c^\top x : x \in P\} < \infty$ and $F = \{x \in P : c^\top x = \delta\}$.

(3) $F = \{x \in P : A'x = b'\} \neq \emptyset$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.

Hint: To show that (2) implies (3) let c be a vector such that $c^\top x$ is finite for every $x \in P$. Then consider the *maximal* subsystem $A'x \leq b'$ of $Ax \leq b$ such that $A'x = b'$ for all $x \in F$.