Algorithms Theory

07 – Binomial Queues

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Priority queues: operations

(Priority) queue *Q*

Data structure for maintaining a set of elements, each having an associated priority from a totally ordered universe. The following operations are supported.

Operations:

Q.initialize(): initializes an empty queue *Q*

Q.isEmpty(): returns true iff *Q* is empty

Q.insert(e): inserts element *e* into *Q* and returns a pointer to the node containing *e*

Q.deletemin(): returns the element of *Q* with minimum key and deletes it

Q.min(): returns the element of *Q* with minimum key

Q.decreasekey(v,k): decreases the value of *v*'s key to the new value *k*

Priority queues: operations

Additional operations:

Q.delete(v): deletes node *v* and its element from *Q*
(without searching for *v*)

Q.meld(Q´): unites *Q* and *Q´* (concatenable queue)

Q.search(k): searches for the element with key *k* in *Q* (searchable queue)

And many more, e.g. *predecessor, successor, max, deletemax*

Priority queues: implementations

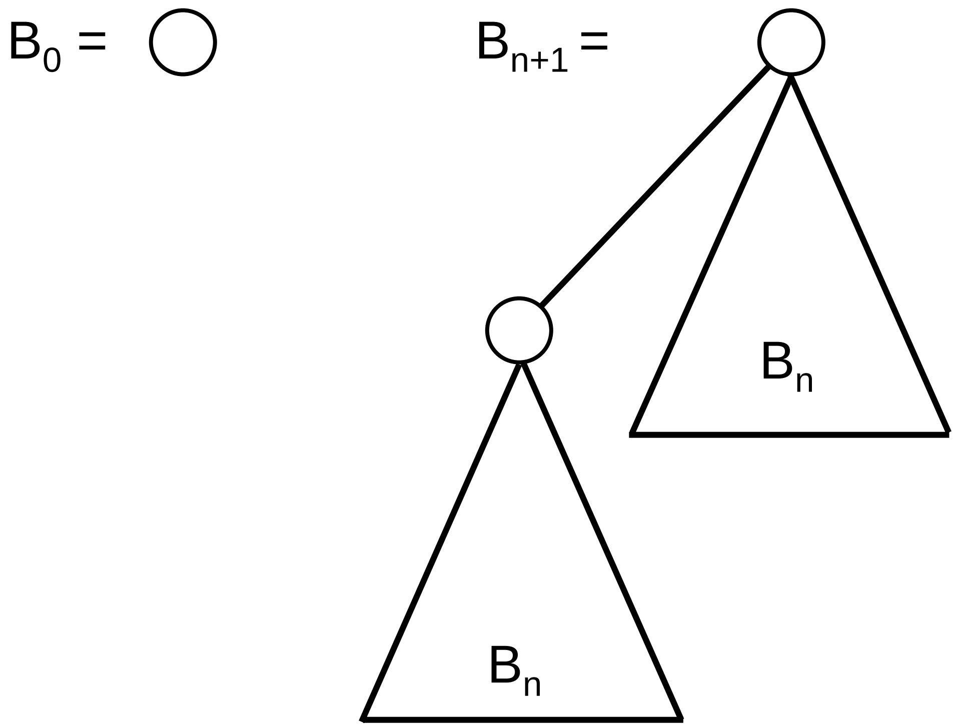
	List	Heap	Bin. – Q.	Fib.-Hp.
insert	O(1)	O(log n)	O(log n)	O(1)
min	O(n)	O(1)	O(log n)	O(1)
delete-min	O(n)	O(log n)	O(log n)	O(log n)*
meld ($m \le n$)	O(1)	O(n) or O(m log n)	O(log n)	O(1)
decr.-key	O(1)	O(log n)	O(log n)	O(1)*

*= amortized cost

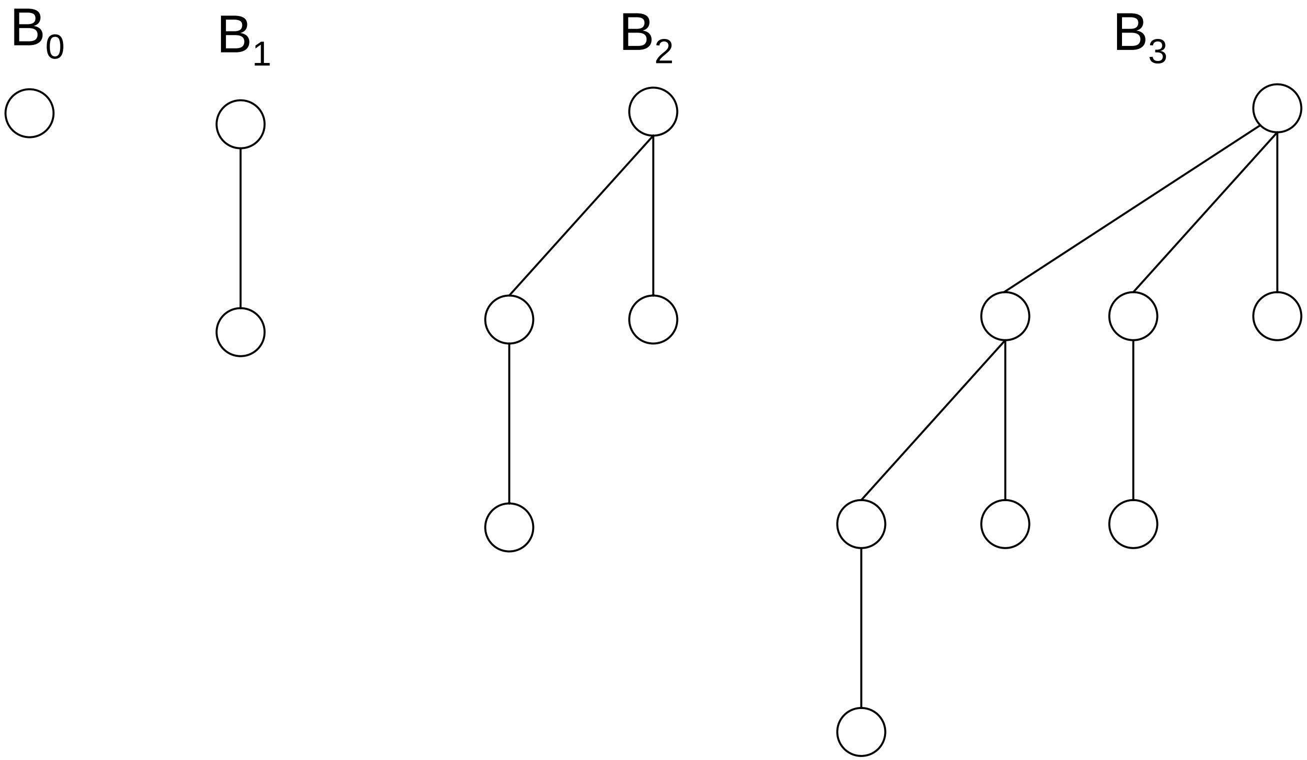
Q.delete(e) = Q.decreasekey(e, -∞) + Q.deletemin( )

Definition

Binomial tree B_n of order *n* $(n \geq 0)$

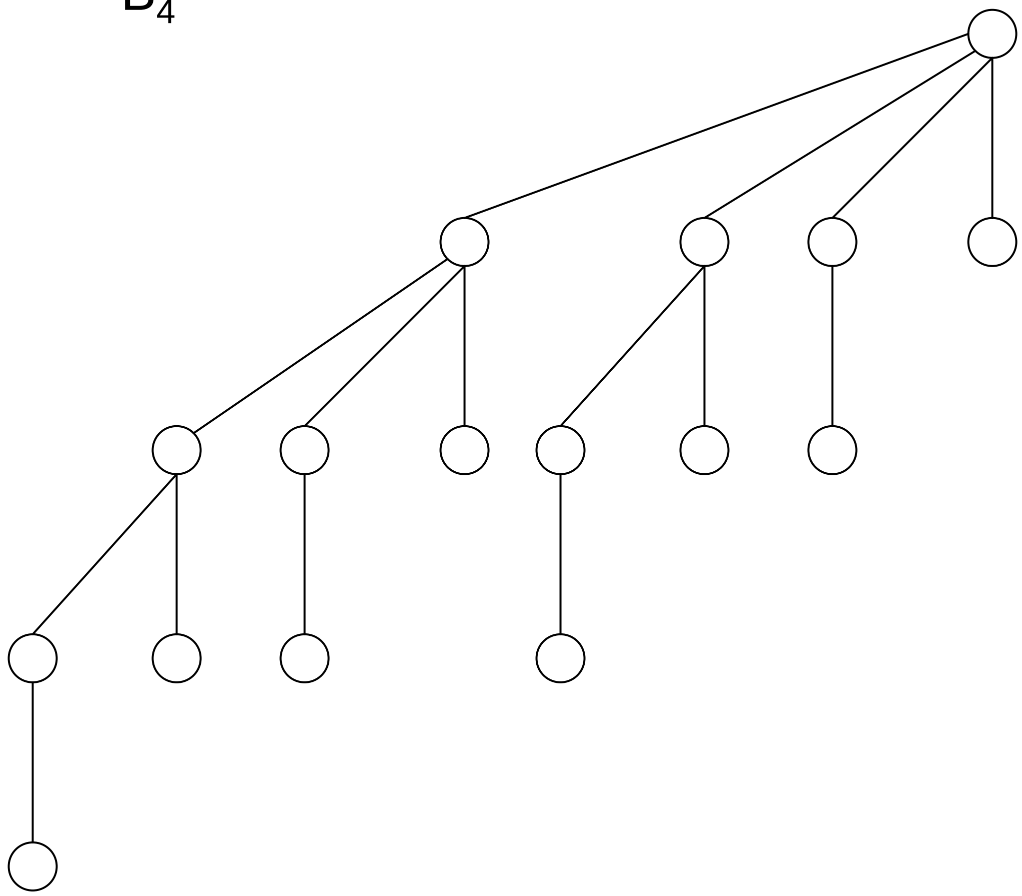


Binomial trees



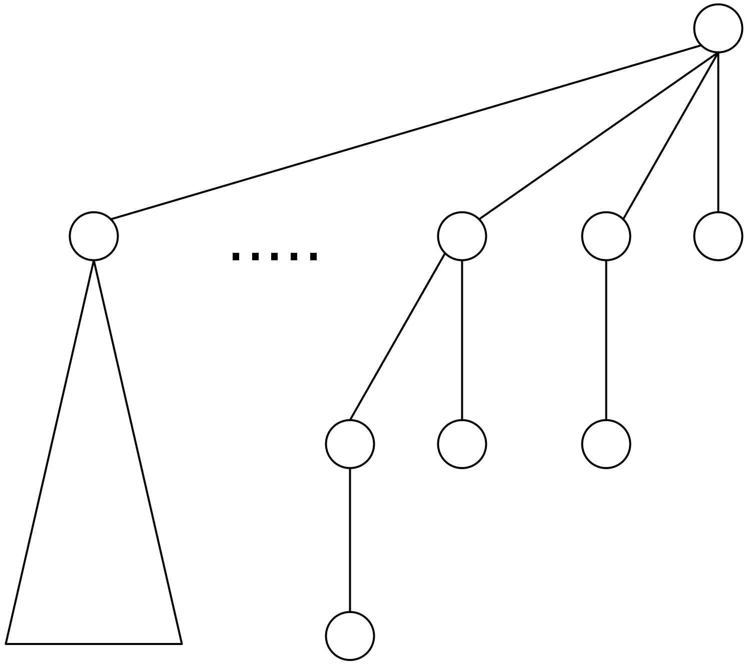
Binomial trees

B_4



Properties

1. B_n contains 2^n nodes.
2. The height of B_n is n.
3. The root of B_n has degree n.
4. B_n =



5. There are exactly $\binom{n}{i}$ nodes at depth i in B_n.

Binomial coefficients

$\binom{n}{i}$ = # *i*-element subsets that can be chosen from an *n*-element set

Pascal's triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Number of nodes at depth *i* in B_n

There are exactly $\binom{n}{i}$ nodes at depth *i* in B_n .

Binomial queues

Binomial queue *Q*:

Set of heap ordered binomial trees of different order to store keys.

***n* keys:**

$B_i \in Q \iff i$-th bit in $(n)_2 = 1$

9 keys:

$\{2, 4, 7, 9, 12, 23, 58, 65, 85\}$

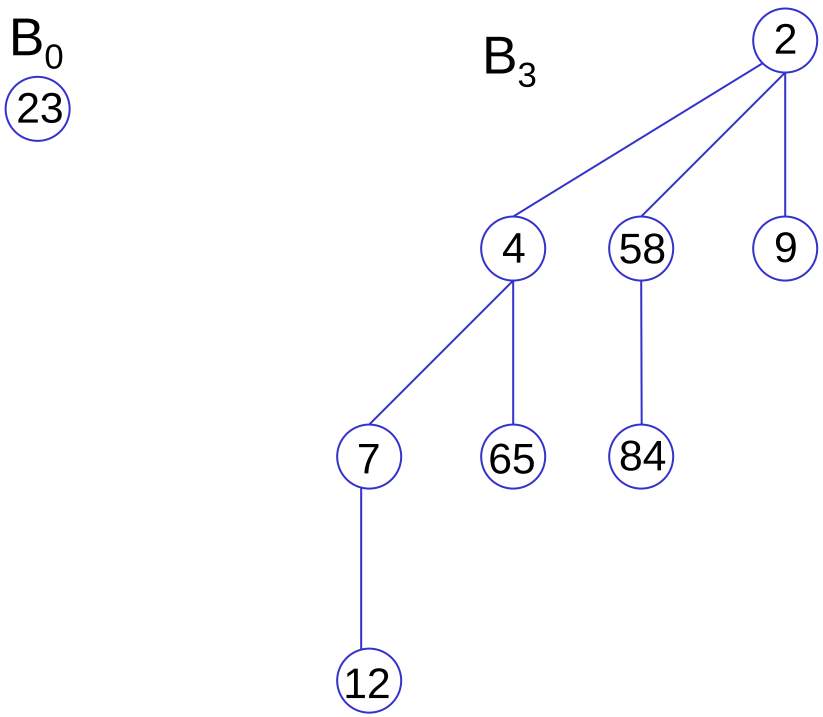
$9 = (1001)_2$

Binomial queues: 1st example

9 keys:

{2, 4, 7, 9,12, 23, 58, 65, 85}

$9 = (1001)_2$



Min can be determined in $O(\log n)$ time.

Binomial queues: 2nd example

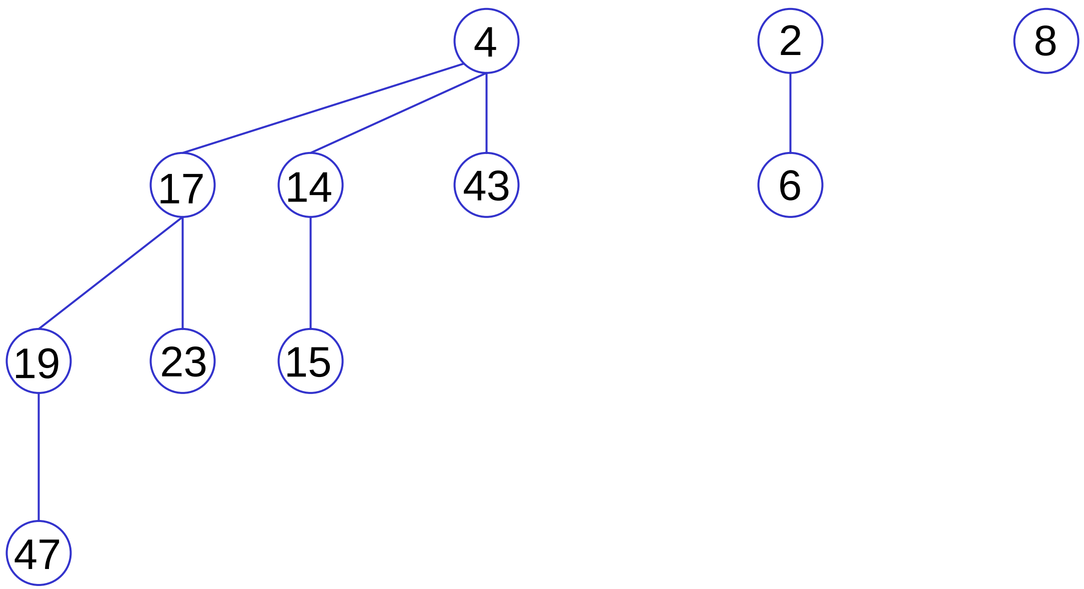
11 keys:

{2, 4, 6, 8, 14, 15, 17, 19, 23, 43, 47}

$11 = (1011)_2 \rightarrow$ 3 binomial trees

B_3, B_1 and B_0

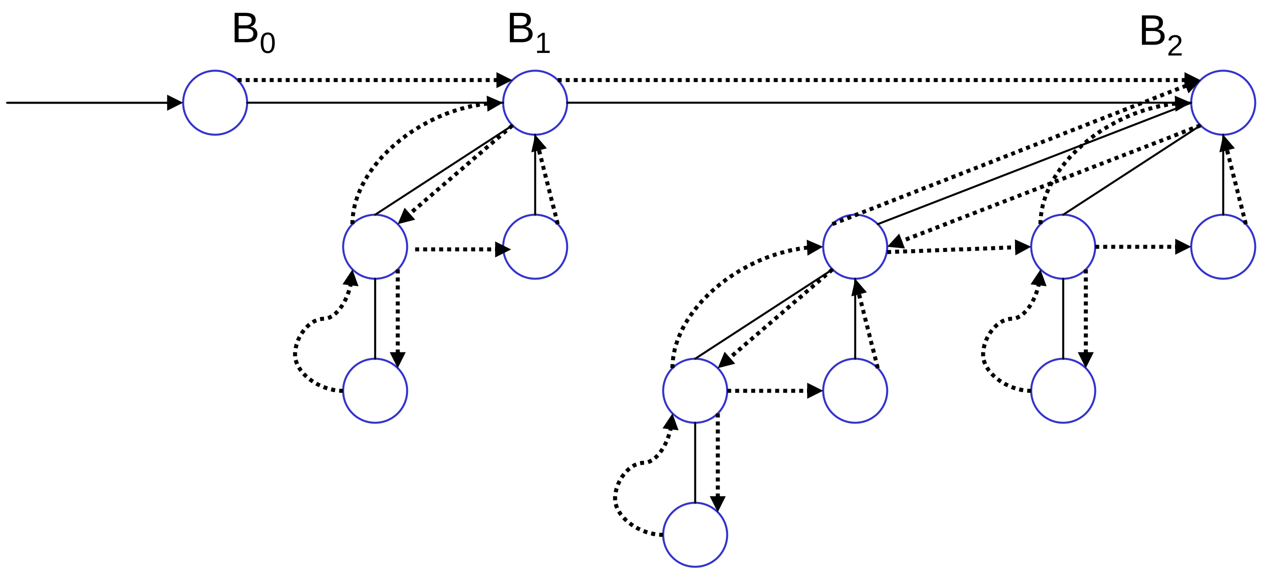
Q_{11}:



Child-sibling representation

Structure of a node:

parent	
key	degree
child	sibling



Binomial trees: operation ‘meld’ (‘link’)

Unite two binomial trees *B, B´* of the same order

$$B_n + B_n \rightarrow B_{n+1}$$

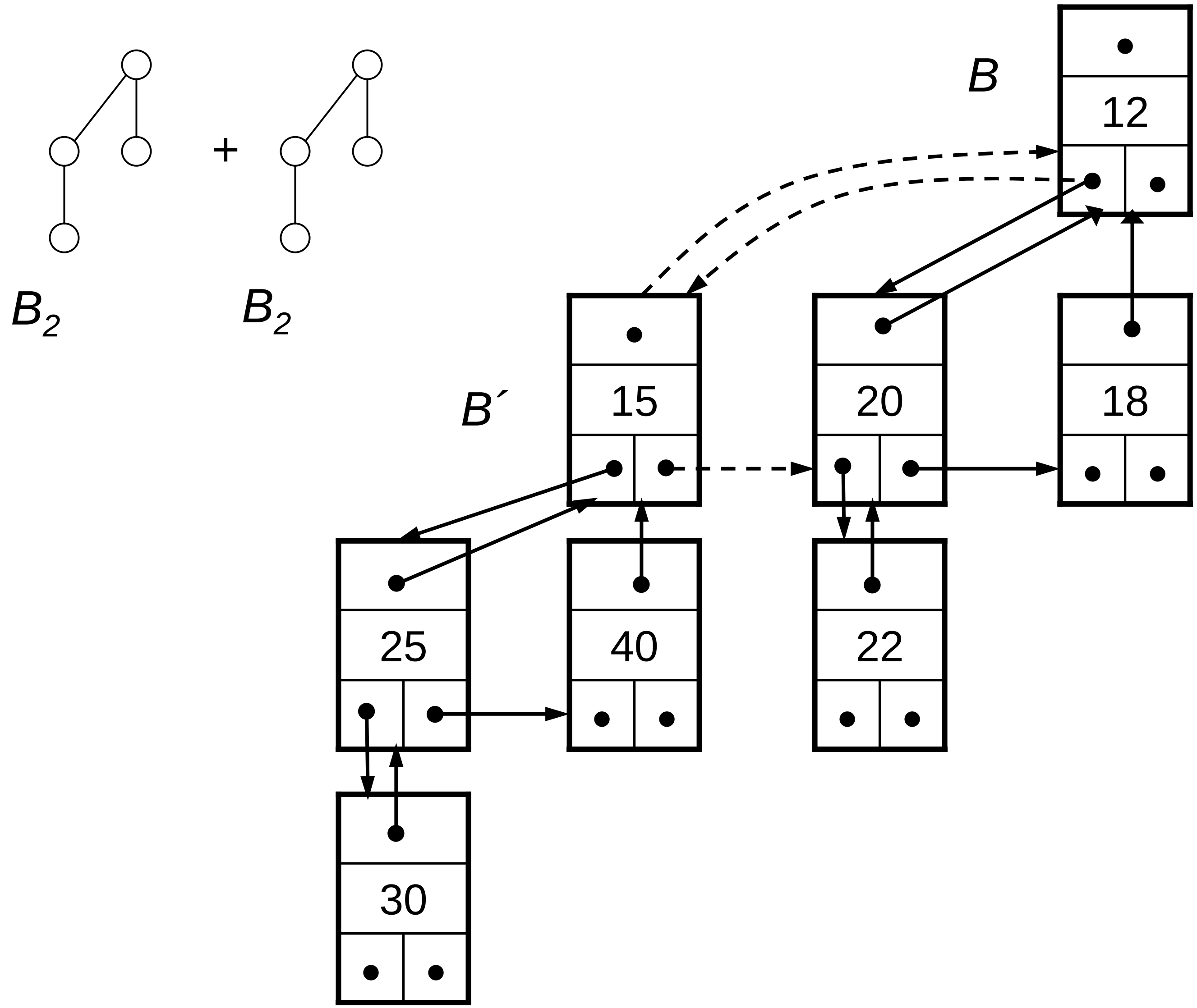
procedure Link:

B.Link(B´)

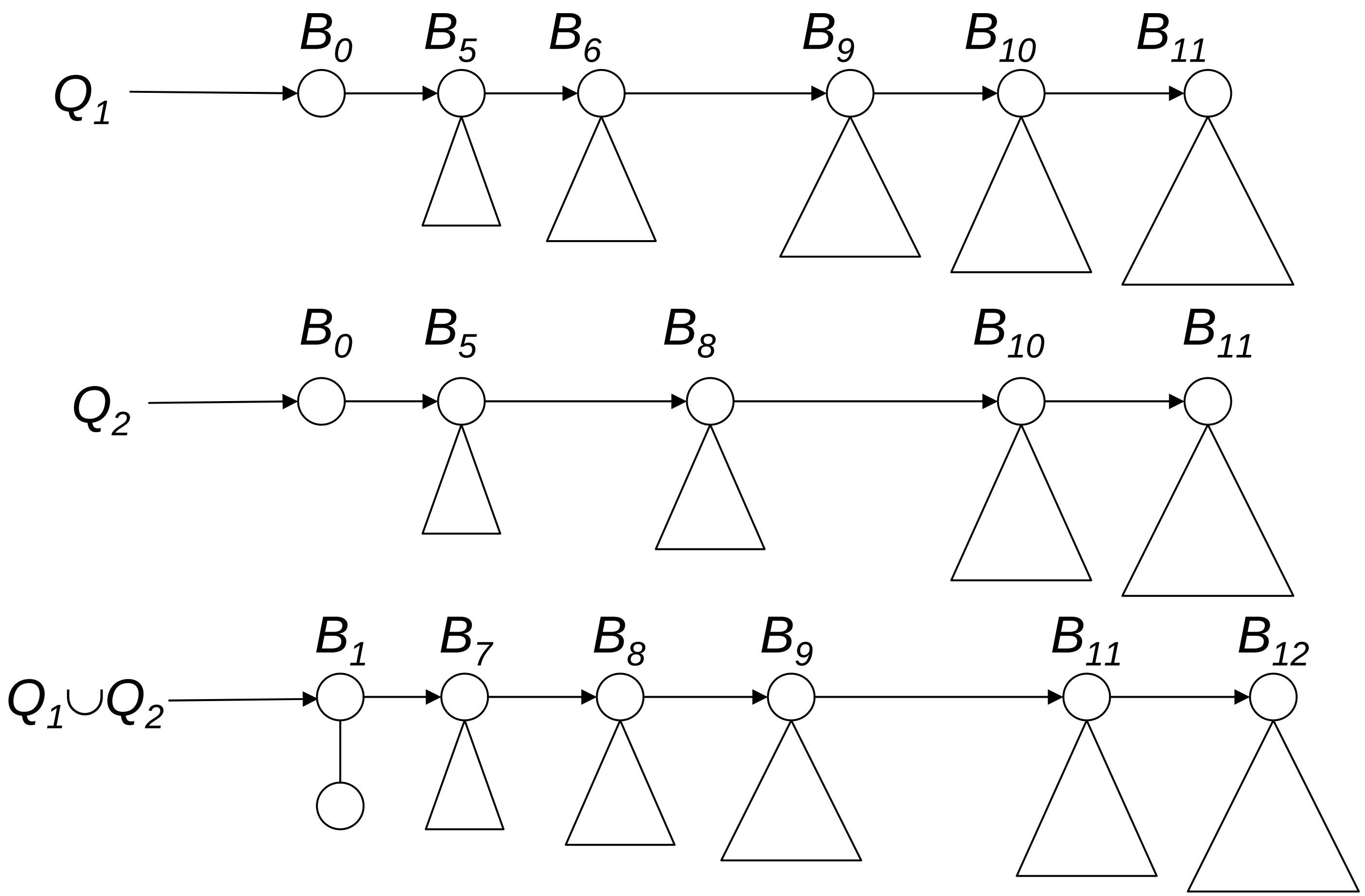
```
/* Make the root with the larger key a child of the root with the smaller key. */
if B.key >  B´.key
   then B´.Link(B)
          return
   /* B.key ≤ B´.key*/
B´.parent = B
B´.sibling = B.child
B.child = B´
B.degree = B.degree +1
```

Running time O(1)

Example of the operation ‘link’



Binomial queues: operation 'meld'



If the operation yields a B_i and the initial lists both contain a B_i, then unite the initial B_i's. Running time: O (log *n*)

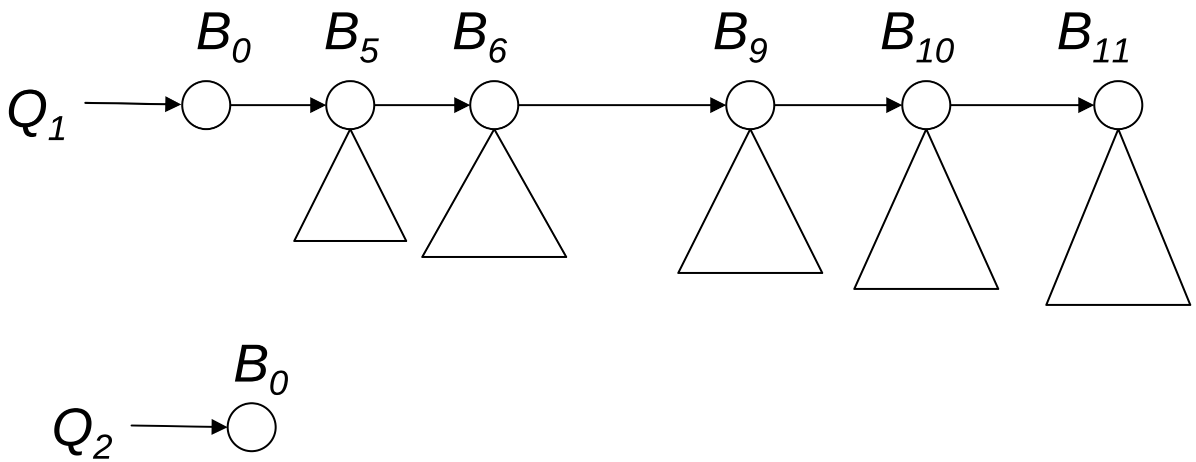
Binomial queues: operations

Q.initialize:

Q.root = null

Q.insert(e):

new B_0
B_0.*key* = e
Q.*meld*(B_0)



Running time: O(log *n*)

Binomial queues: 'deletemin'

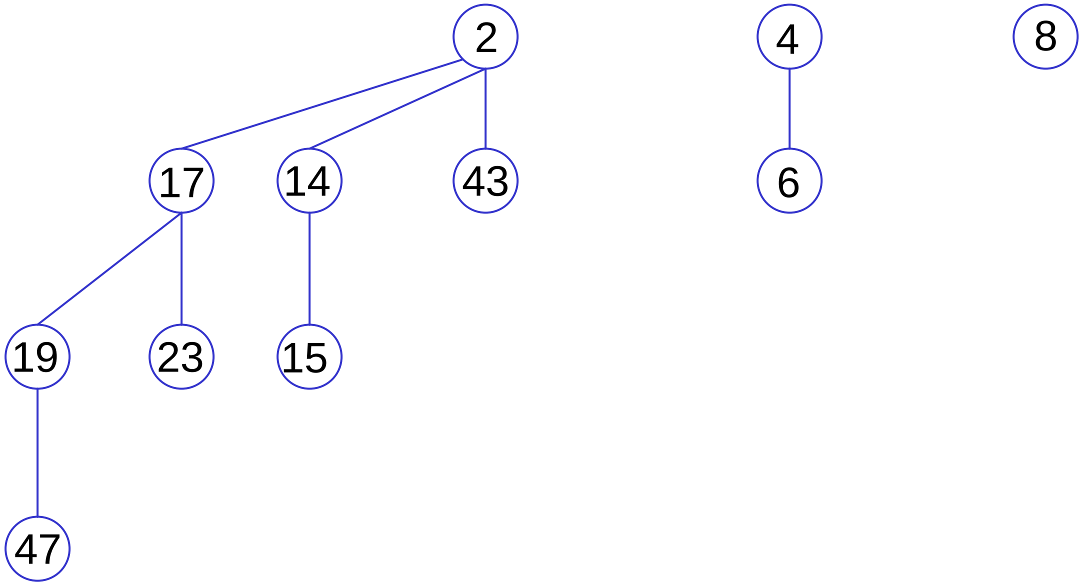
Q.deletemin():

1. Determine B_i whose root has the minimum key in the root list and delete B_i from Q (returns Q´)
2. Insert the children of B_i in reverse order into a new queue : B_0 , B_1 , ... , $B_{i-1} \rightarrow Q''$
3. *Q´.meld(Q´´)*

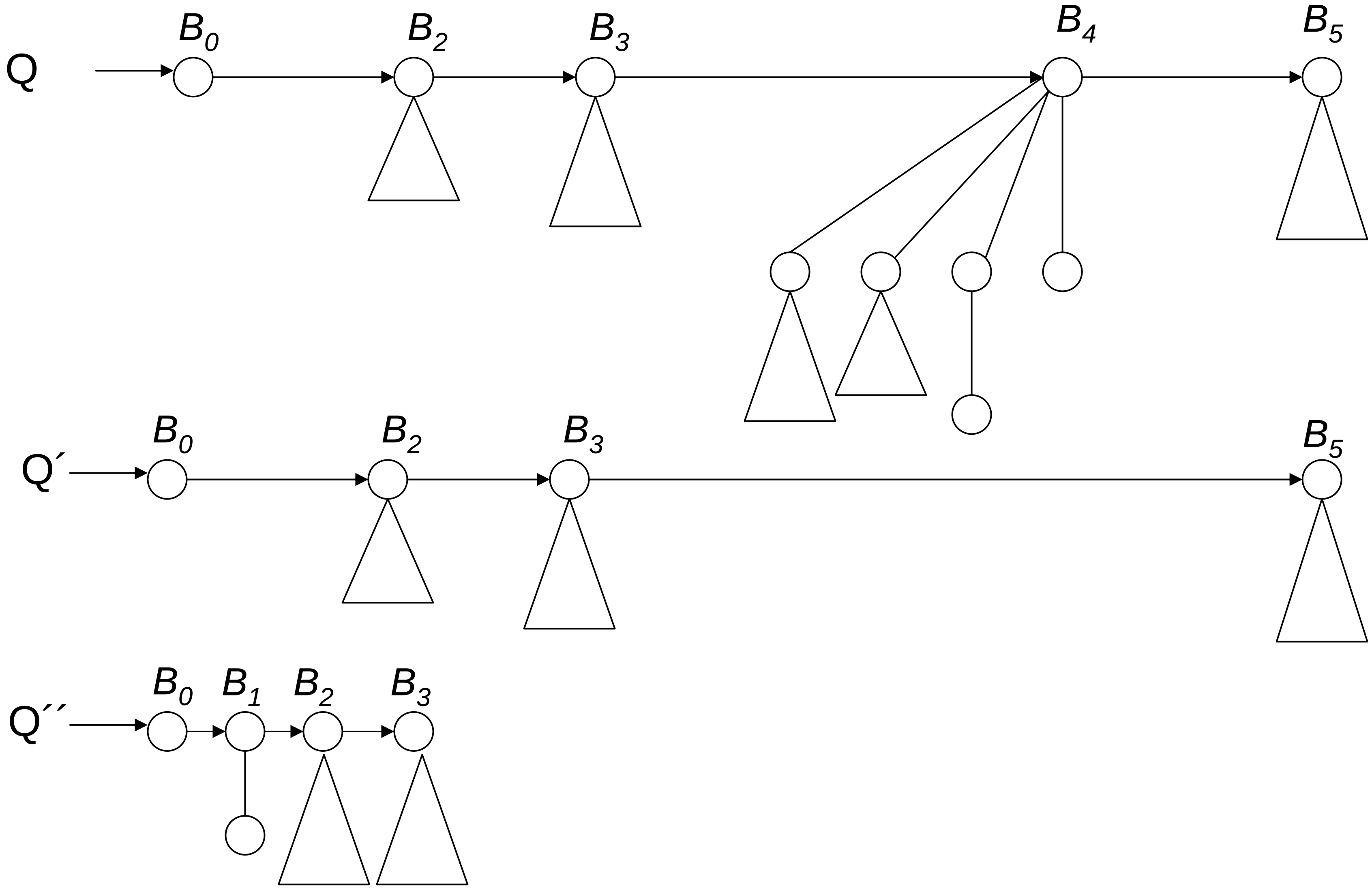
Running time: O(log n)

Binomial queues: ‘deletemin’, 1st example

Q_{11}:



Binomial queues: 'deletemin', 2nd example

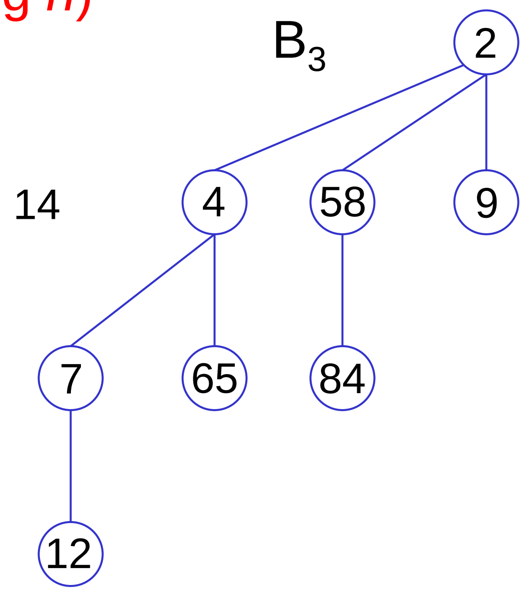


Binomial queues: ‘decreasekey’

Q.decreasekey(v, k):

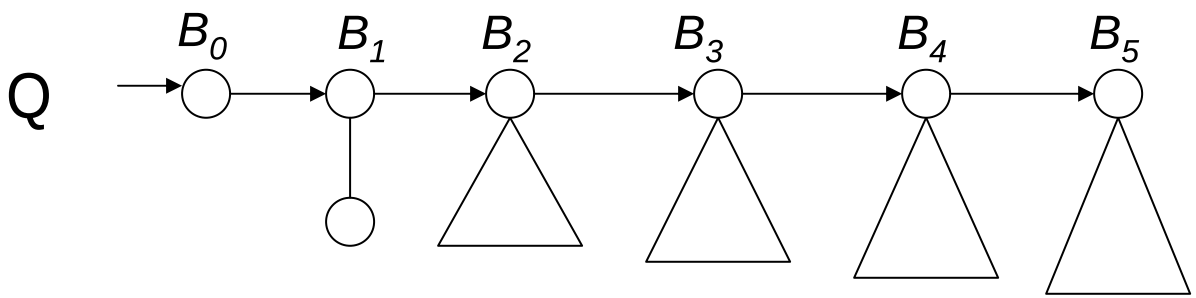
1. *v.element.key := k*
2. Repeatedly exchange *v.element* with the element of *v*‘s parent, until the heap property is restored.

Running time: O (log *n*)

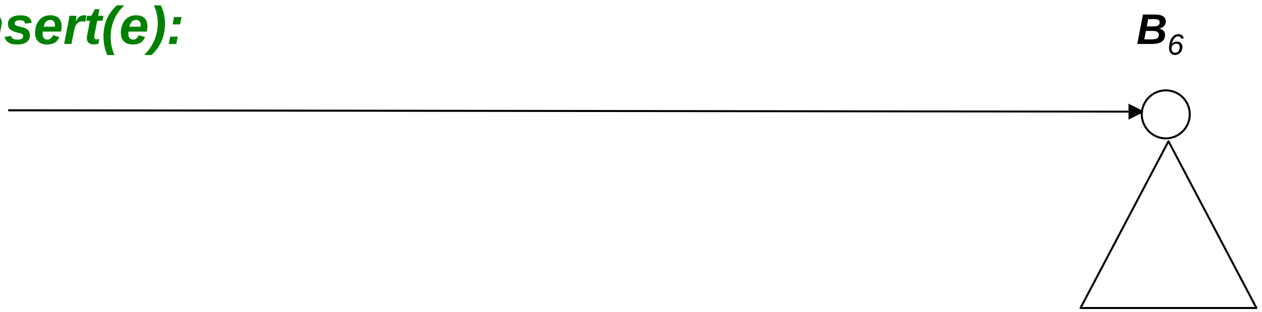


Binomial queues: worst case sequence of operations

Q.deletemin():



Q.insert(e):



Running time:
O(log n)