Algorithms Theory

15 – Text Search (2)

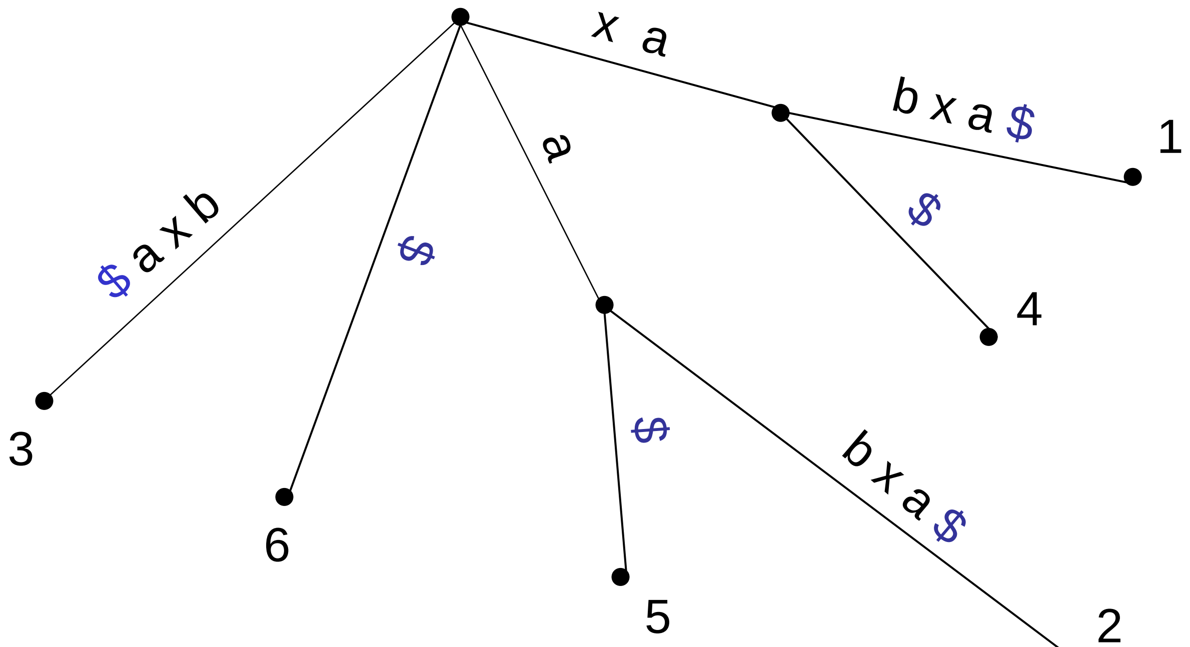
Construction of suffix trees

Prof. Dr. S. Albers

Suffix tree

t = x a b x a $

1 2 3 4 5 6



Ukkonen's algorithm: implicit suffix trees

Definition: An *implicit suffix tree* is a tree obtained from the suffix tree for *t*$ by

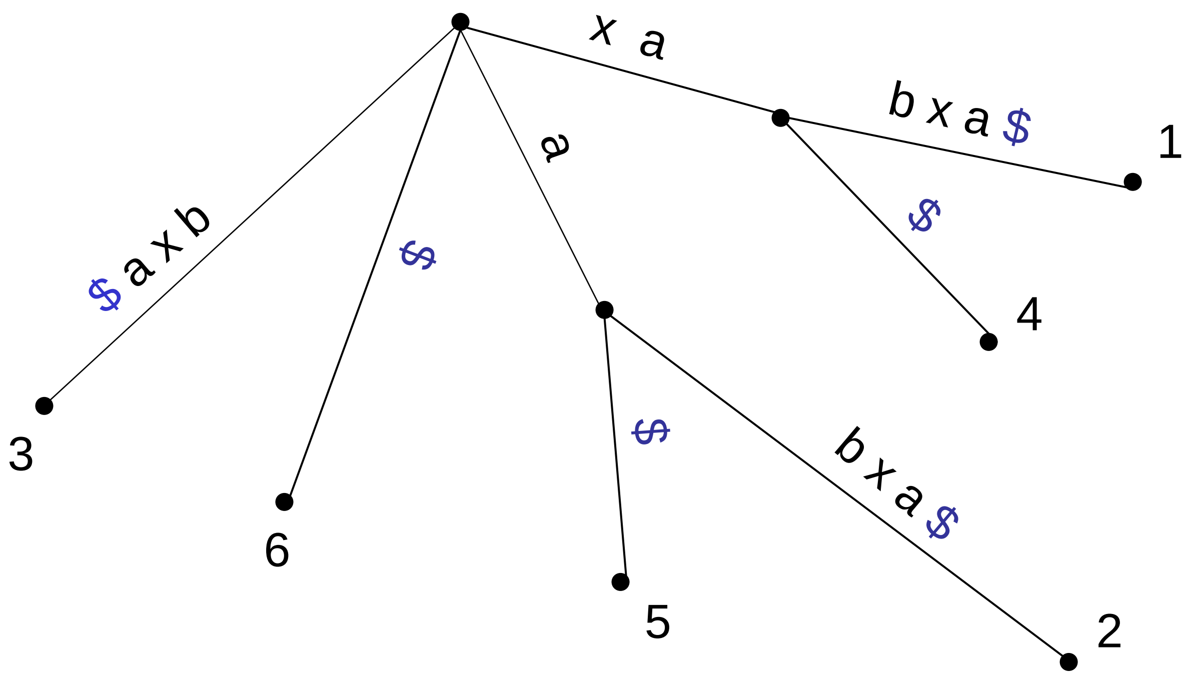
(1) deleting every copy of $ from the edge labels,

(2) deleting edges that have no label,

(3) deleting unary nodes.

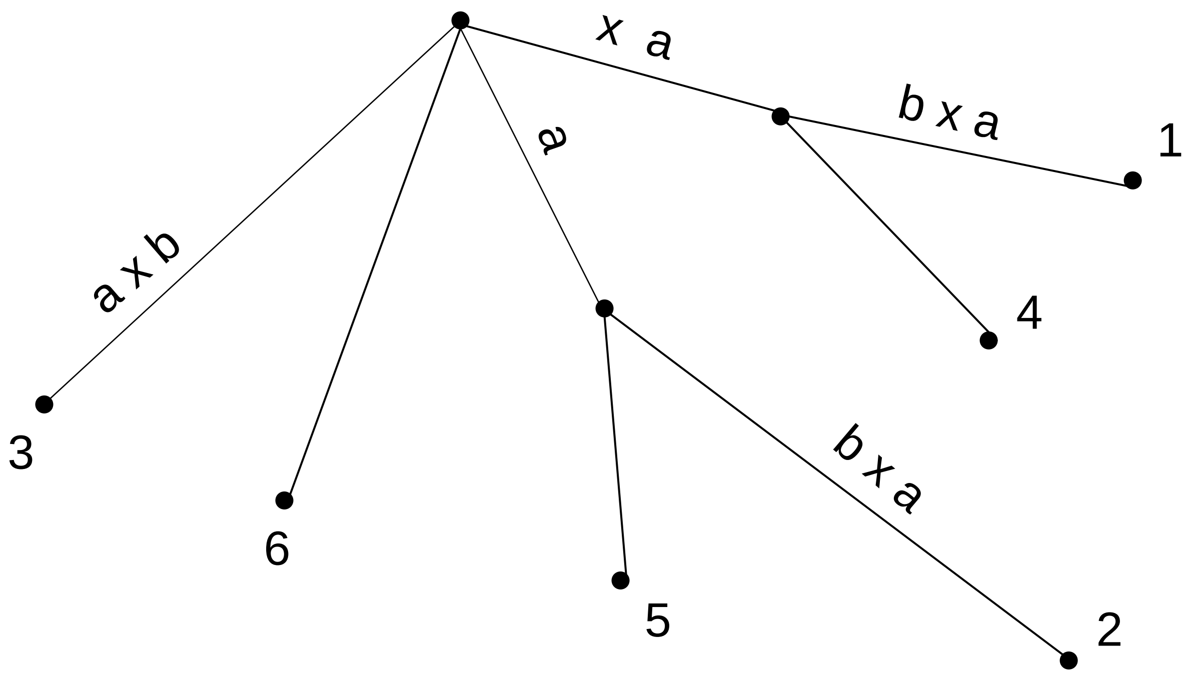
Ukkonen's algorithm: implicit suffix trees

t = x a b x a $
1 2 3 4 5 6



Ukkonen’s algorithm: implicit suffix trees

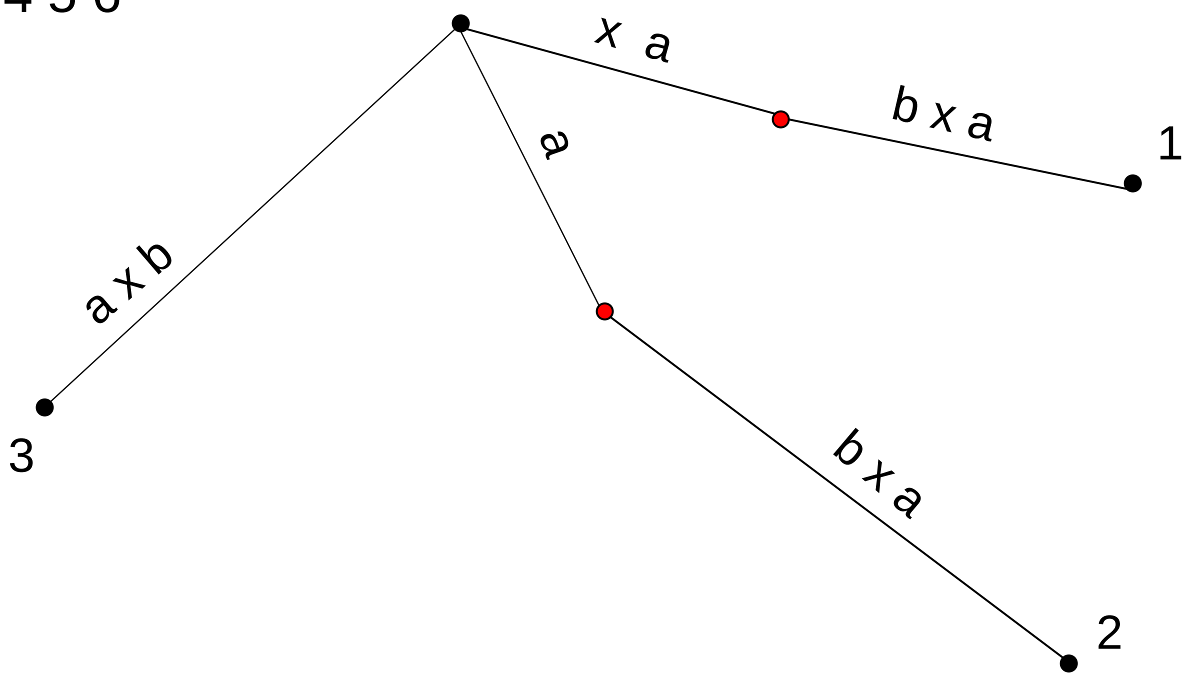
(1) deleting $ from the edge labels



Ukkonen's algorithm: implicit suffix trees

(2) deleting edges that have no label

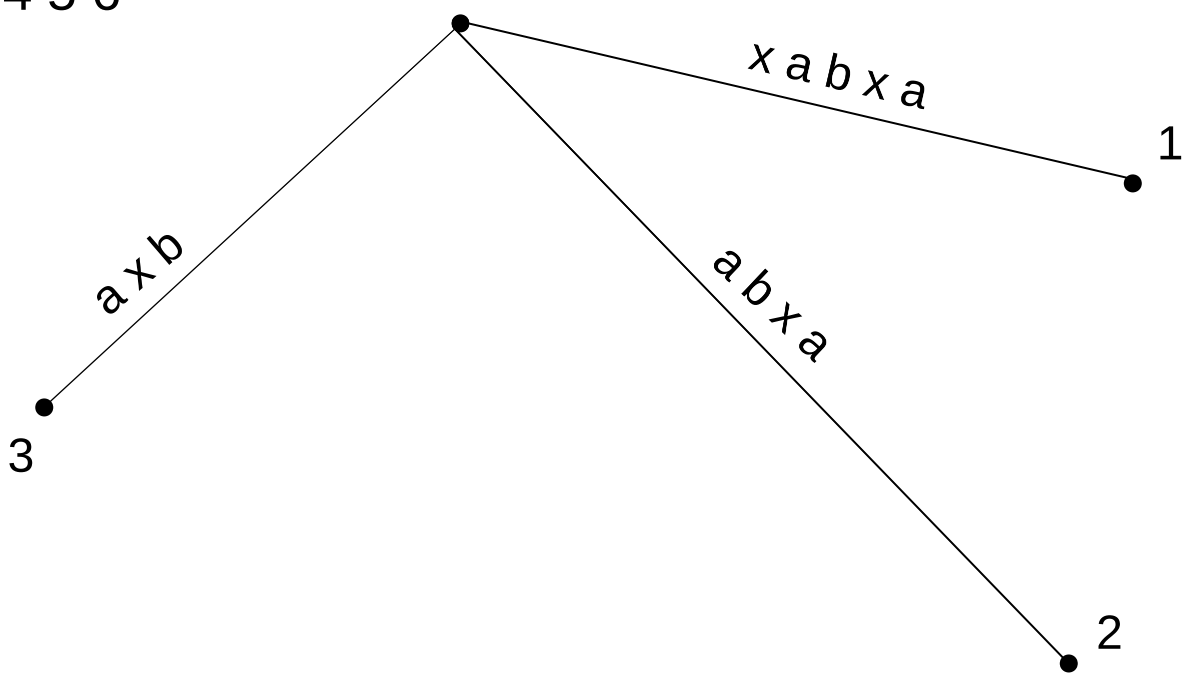
t = x a b x a $
1 2 3 4 5 6



Ukkonen's algorithm: implicit suffix trees

(3) deleting unary nodes

t = x a b x a $
1 2 3 4 5 6



Ukkonen's algorithm

Let $t = t_1 t_2 t_3 \ldots t_m$.

Ukk is an online algorithm: The suffix tree $ST(t)$ is constructed step by step by constructing a sequence of implicit suffix trees for the prefixes of t:

$$ST(\varepsilon),\ ST(t_1),\ ST(t_1 t_2),\ \ldots,\ ST(t_1 t_2 \ldots t_m)$$

$ST(\varepsilon)$ is the empty implicit suffix tree, consisting of the root only.

Ukkonen's algorithm

This is an *online* approach in the sense that in each step, the implicit suffix tree for a prefix of *t* is created without knowledge of the rest of the input string *t*.

Since the algorithm reads the input string character by character from left to right, it works *incrementally*.

Ukkonen's algorithm

Incremental construction of an implicit suffix tree:

Induction basis: $ST(\varepsilon)$ consists of the root only.

Induction step: $ST(t_1 \dots\dots t_i)$ is extended to $ST(t_1 \dots t_i t_{i+1})$ for all $i < m$.

Let T_i be the implicit suffix tree for $t[1...i]$.

- At first, we construct T_1: This tree has a single edge labeled with character t_1.
- In phase i+1, we construct tree T_{i+1} from T_i.
- We iterate for $i = 1 \dots m-1$.

Ukkonen's algorithm

Pseudo code for Ukk:

Construct tree T_1.

for *i* = 1 **to** *m*–1 **do**

begin {phase *i*+1}

    for *j* = 1 **to** *i* +1 **do**

        begin {extension *j*}

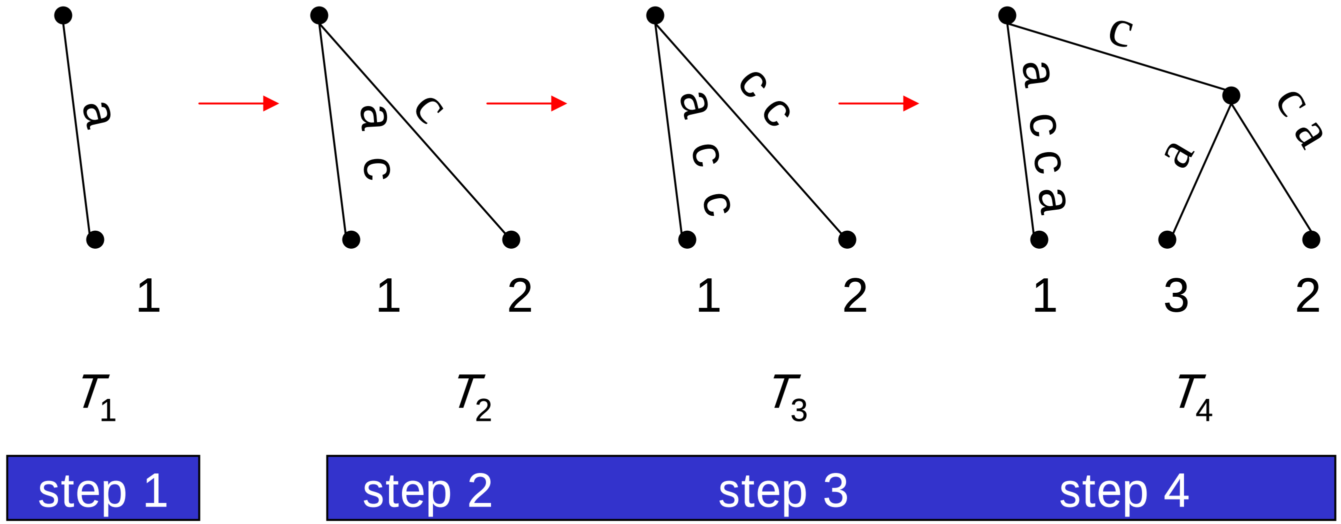
            In the current tree find the end of the path from the root labeled *t*[*j* ... *i*]. If necessary, extend that path by adding character *t*[*i*+1], thus ensuring that string *t*[*j*...*i*+1] is in the tree.

        end;

end;

Ukkonen's algorithm

t = a c c a $



Ukkonen's algorithm

- In extension *j* of phase *i+1*, the end of the path from the root labeled with substring *t*[*j*...*i*] is determined. Then, this substring is extended by adding the character *t*[*i*+1] to its end (unless *t*[*i*+1] already appears there).

- In phase *i*+1, string *t*[1...*i*+1] is first inserted into the tree, followed by strings *t*[2...*i*+1] , *t*[3...*i*+1] ,.... (in extensions 1,2,3,...., respectively).

- Extension *i*+1 of phase *i*+1 inserts the single character string *t*[*i*+1] into the tree (unless it is already there).

Ukk: Suffix extension rules

Extension *j* (in phase *i*+1) results from applying one of the following rules:

Rule 1: If the path *t* [*j*...*i*] ends at a leaf, character *t* [*i*+1] is added to the end of the label on that leaf edge.

Rule 2: If no path from the end of string *t* [*j*...*i*] starts with character *t* [*i*+1], then a new leaf edge labeled with character *t* [*i*+1] is created. A new internal node will also be created there if *t* [*j*...*i*] ends inside an edge. (This is the only extension that increases the number of leaves! The new leaf represents the suffix starting at position *j*.)

Rule 3: If some path from the end of string *t* [*j* ...*i*] starts with character *t* [*i*+1], then string *t* [*j*…*i* +1] is already in the current tree, so we do nothing.

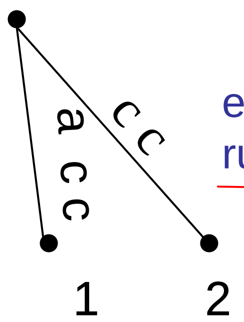
Ukkonen’s algorithm

t = a c c a $

t [1...3] = acc

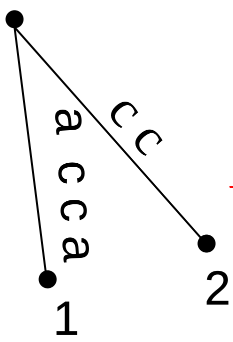
t [1...4] = acca

T_3



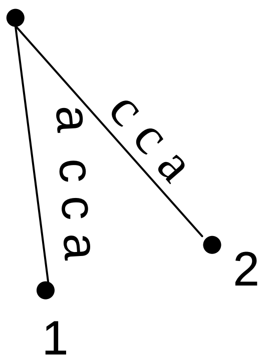
t [1..4] = acca

extend suffix 1
rule 1



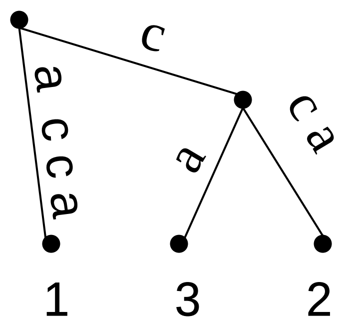
t [2..4] = cca

extend suffix 2
rule 1



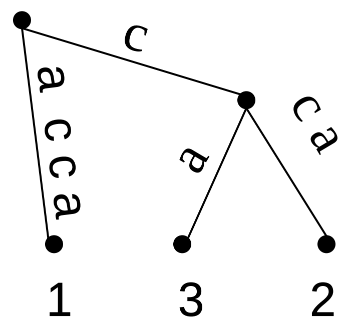
t [3..4] = ca

extend suffix 3
rule 2



t [4..4] = a

a is already in
the tree
rule 3



T_4

Ukkonen's algorithm

During phase i+1 (when T_{i+1} is constructed from T_i) the following holds:

(1) If rule 3 applies in extension j, then the path labeled $t\,[j...i]$ in T_i must continue with character $t\,[i+1]$. So, any path labeled $t\,[j'...\,i]$ for $j' \geq j$ also continues with character $t\,[i+1]$.

Therefore, rule 3 again applies in extensions $j' = j+1,..., i+1$.

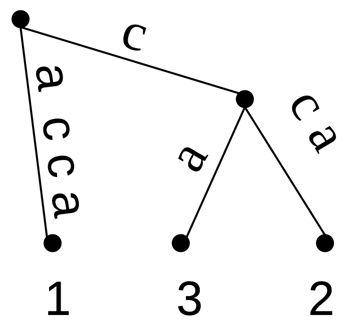
Once rule 3 applies in an extension of phase i+1, this phase may be ended.

Ukkonen’s algorithm

(2) If a leaf is created in T_i, then it will remain a leaf in all successive trees $T_{i'}$ for $i' > i$ (once a leaf, always a leaf!).

Reason: A leaf edge is never extended beyond its current leaf.

t = a c c a b a a c b a ....



T_4

Ukkonen's algorithm

Implication:

- Leaf 1 is created in phase 1. In each phase i+1 there is an initial sequence of successive extensions (starting with extension 1) where rule 1 or 2 applies.

- Let j_i denote the last extension in this sequence of phase i.

  Then: $j_i \leq j_{i+1}$

Ukkonen's algorithm

Extensions according to rule 1 may be performed implicitly!

Ukkonen's algorithm

Improving the algorithm:

In phase i+1, rule 1 applies in all extensions j for $j \in [1, j_i]$.
Only constant time is required to do those extensions implicitly.

If $j \in [j_i +1, i+1]$, then find the end of the path labeled $t[j \ldots i]$ and extend it by character t[i+1] according to rules 2 or 3.
If rule 3 applies, set $j_{i+1} = j$ -1 and end phase i+1.

Ukkonen's algorithm

Example:

phase 1:	compute extensions	$1 \ldots j_1$
phase 2:	compute extensions	$j_1 + 1 \ldots j_2$
phase 3:	compute extensions	$j_2 + 1 \ldots j_3$
....		
phase i-1:	compute extensions	$j_{i-2} + 1 \ldots j_{i-1}$
phase i:	compute extensions	$j_{i-1} + 1 \ldots j_i$

Ukkonen's algorithm

- As long as explicit extensions are performed, keep track of the index j^* of the current explicit extension.
- During the execution of the algorithm, j^* never decreases.
- As there are only m phases (where $m = |t|$) and j^* is bounded by m, the algorithm performs only m explicit extensions.

Ukkonen's algorithm

Extended pseudo code for Ukk:

Construct tree T_1; $j_1 = 1$;
for $i = 1$ **to** $m - 1$ **do**
begin {phase i+1}
  Do all implicit extensions.
  for $j = j_i + 1$ **to** $i + 1$ **do**
    begin {extension j}
    In the current tree find the end of the path from the root labeled $t[j \dots i]$. If necessary, extend that path by adding character $t[i+1]$, thus ensuring that string $t[j \dots i+1]$ is in the tree.
    $j_{i+1} := j$;
    if rule 3 was applied **then** $j_{i+1} := j - 1$ and phase i+1 ends;
    end;
end;

Ukkonen's algorithm

t = pucupcupu

i:	0	1	2	3	4	5	6	7	8	9
	ε	***p**	pu	puc	pucu	pucup	pucupc	pucupcu	pucupcup	pucupcupu
			***u**	uc	ucu	ucup	ucupc	ucupcu	ucupcup	ucupcupu
				***c**	cu	cup	cupc	cupcu	cupcup	cupcupu
					u	***up**	upc	upcu	upcup	upcupu
						p	***pc**	pcu	pcup	pcupu
							c	**cu**	**cup**	***cupu**
								u	up	***upu**
									p	**pu**
										u

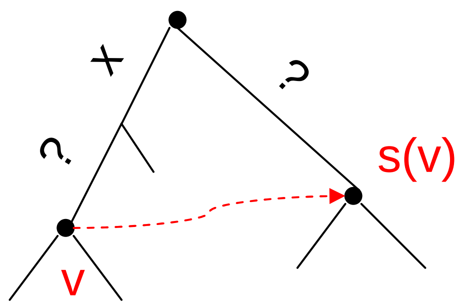
- Suffixes that cause an extension according to rule 2 are marked with *.
- Underlined suffixes indicate the last extension where rule 2 applies.
- Suffixes that end a phase (the first time rule 3 applies) are colored blue.

Ukkonen's algorithm

The running time may be improved using suffix links.

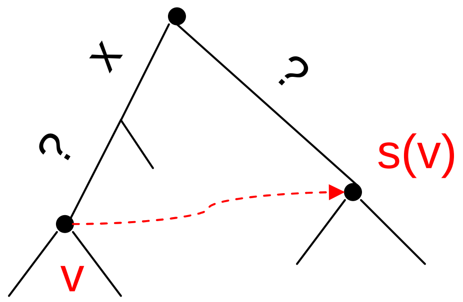
Definition: Let *x*? be an arbitrary string where *x* is a single character and ? some (possibly empty) substring.
For an internal node *v* with edge labels *x*? the following holds:

> If there exists a node *s*(*v*) with edge label ?, then there is a pointer from *v* to *s*(*v*) which is called a suffix link.



Ukkonen's algorithm

Idea:
By following the suffix links, we do not have to start each search for a split point at the root node. Instead, we can use the suffix links in order to determine these nodes more efficiently, i.e. in constant amortized time.



Ukkonen's algorithm

- Using suffix links, extension rules 2 and 3 can be applied more efficiently.
- Any explicit extension takes amortized O(1) time (not shown here).
- Since there are only *m* explicit extensions, the total running time of Ukkonen's algorithm is O(*m*) (where $m = |t|$).

Ukkonen's algorithm

The true suffix tree:

The final implicit suffix tree T_m can be converted to a true suffix tree in O(m) time.

(1) Add a terminal symbol $ to the end of *t*.

(2) Let Ukkonen's algorithm continue with this character.

The resulting tree is the true suffix tree where no suffix is prefix of another suffix and where each suffix ends at a leaf.