Algorithms Theory

12 – Minimum Spanning Trees

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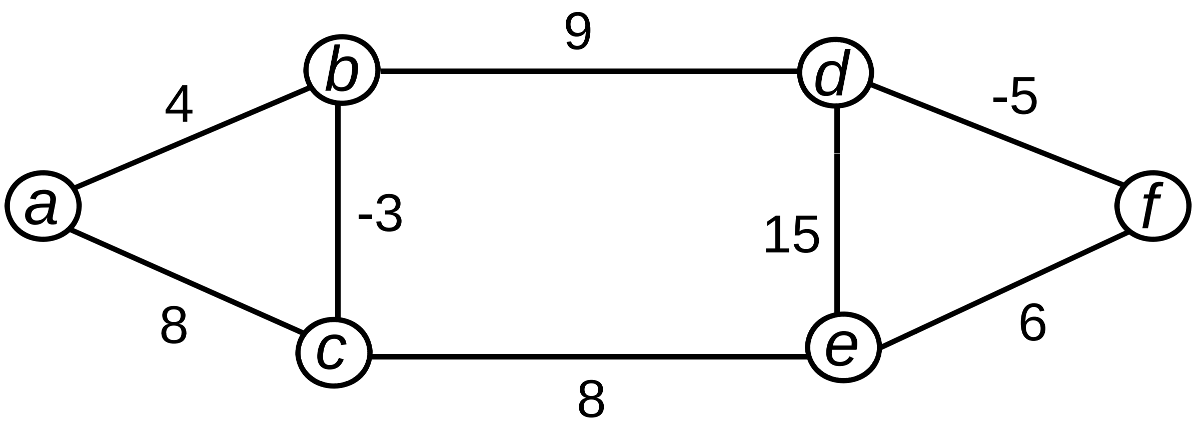
1. Minimum spanning trees

$G = (V, E)$ undirected graph $\qquad w: E \to R$ weight function

Let $T \subseteq E$ be a tree (connected, acyclic subgraph).

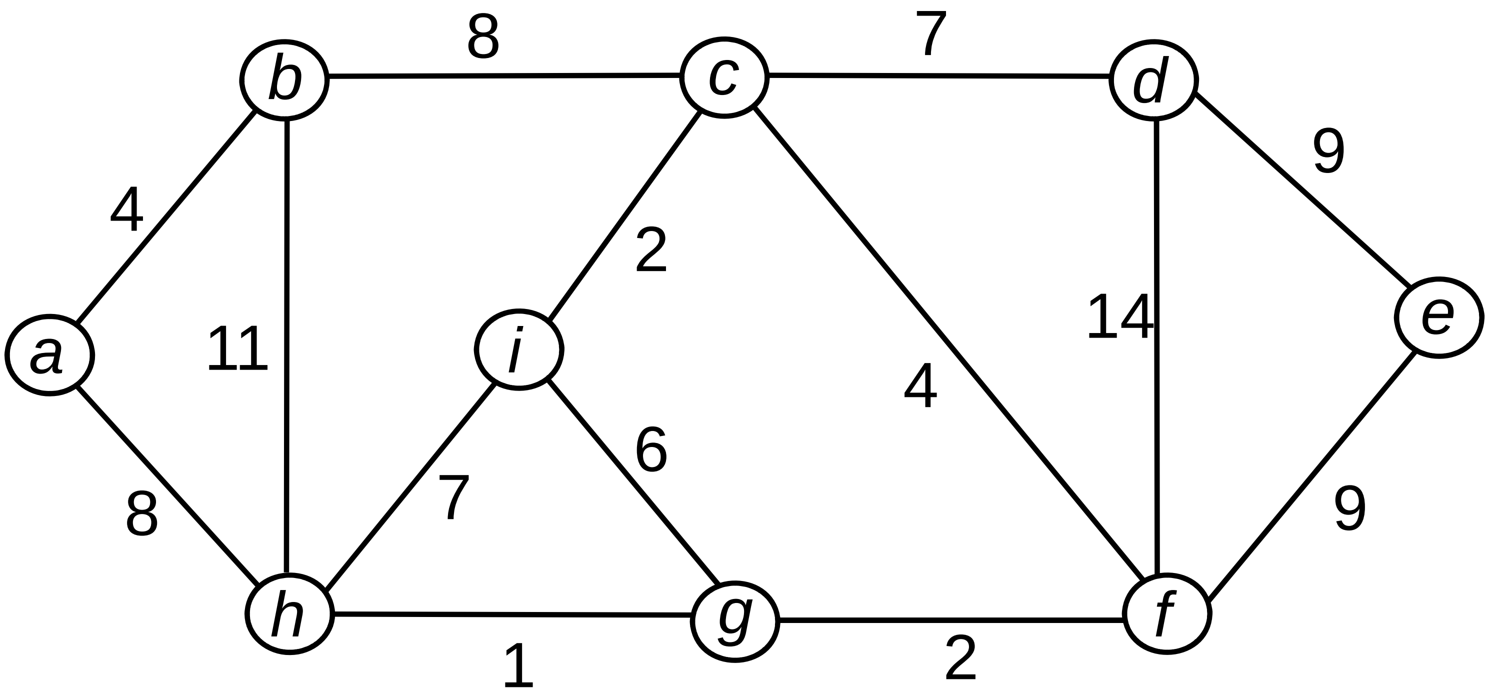
Total weight of T :

$$w(T) = \sum_{(u,v) \in T} w(u,v)$$



Minimum spanning trees

A tree $T \subseteq E$ that connects all vertices in V and whose total weight is minimal is called a minimum spanning tree.



Growing a minimum spanning tree

Invariant: Maintain a set $A \subseteq E$ that is a subset of some minimum spanning tree.

Definition: An edge $(u,v) \in E \setminus A$ is a safe edge for A if $A \cup \{(u,v)\}$ is also a subset of some minimum spanning tree.

Greedy approach

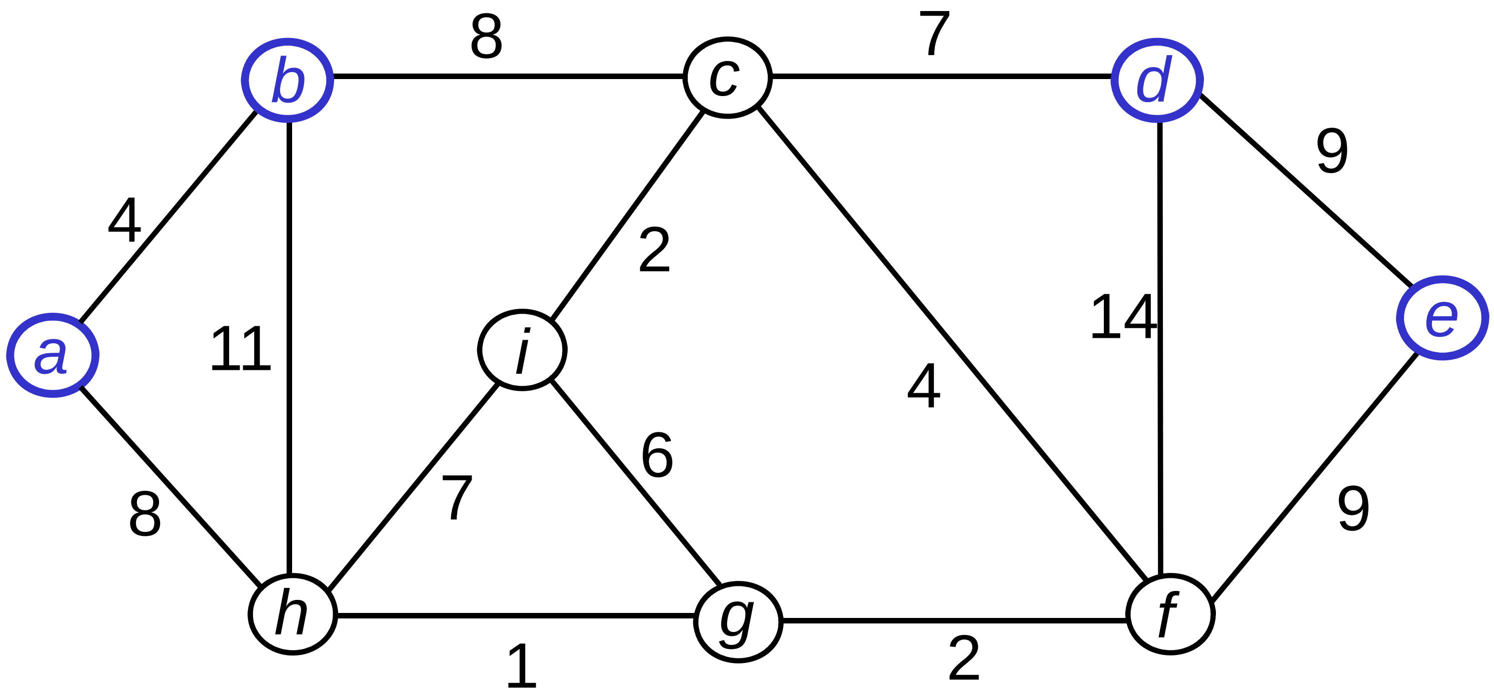
Algorithm Generic-MST(G,w);

1. $A \leftarrow \emptyset$;
2. **while** A does not form a spanning tree **do**
3. Find an edge (u,v) that is safe for A;
4. $A \leftarrow A \cup \{(u,v)\}$;
5. **endwhile;**

2. Cuts

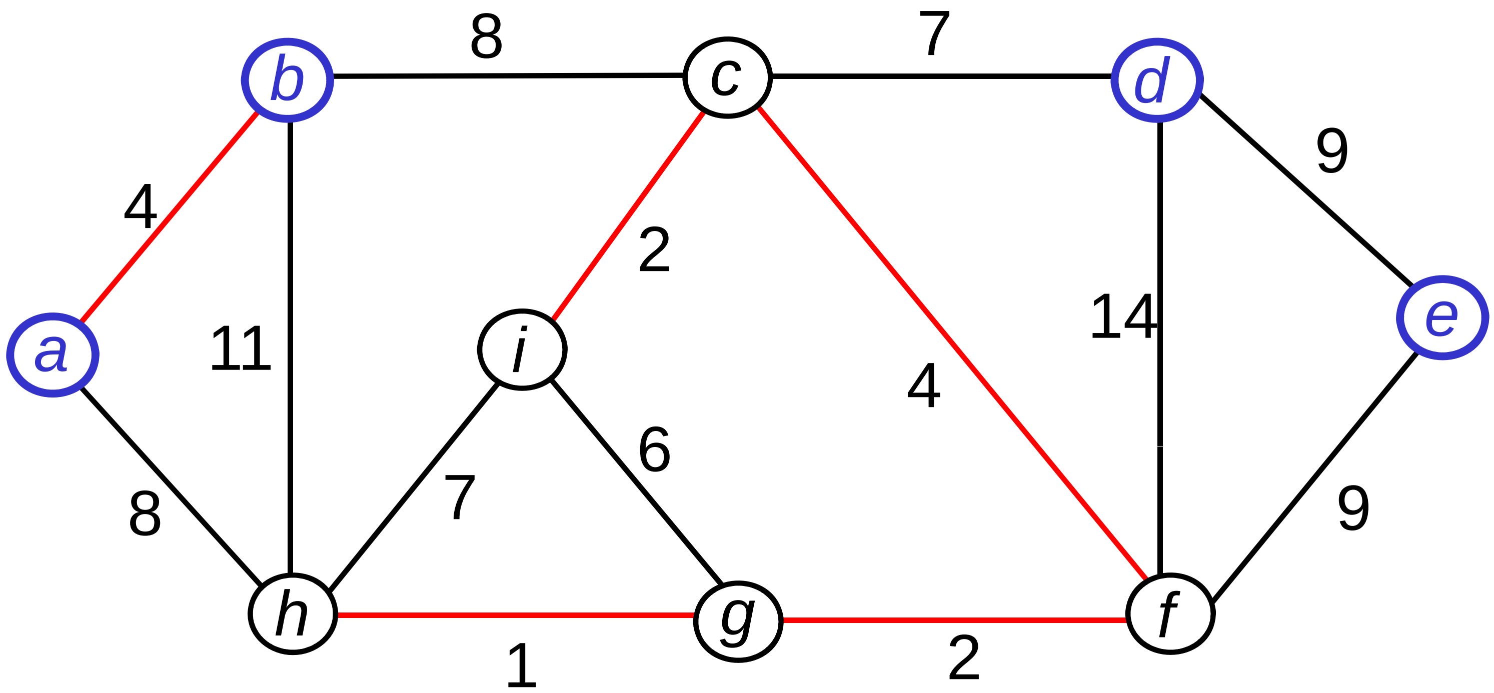
A cut $(S, V \setminus S)$ is a partition of V.

An edge (u,v) crosses $(S, V \setminus S)$ if one of its endpoints is in S and the other is in $V \setminus S$.



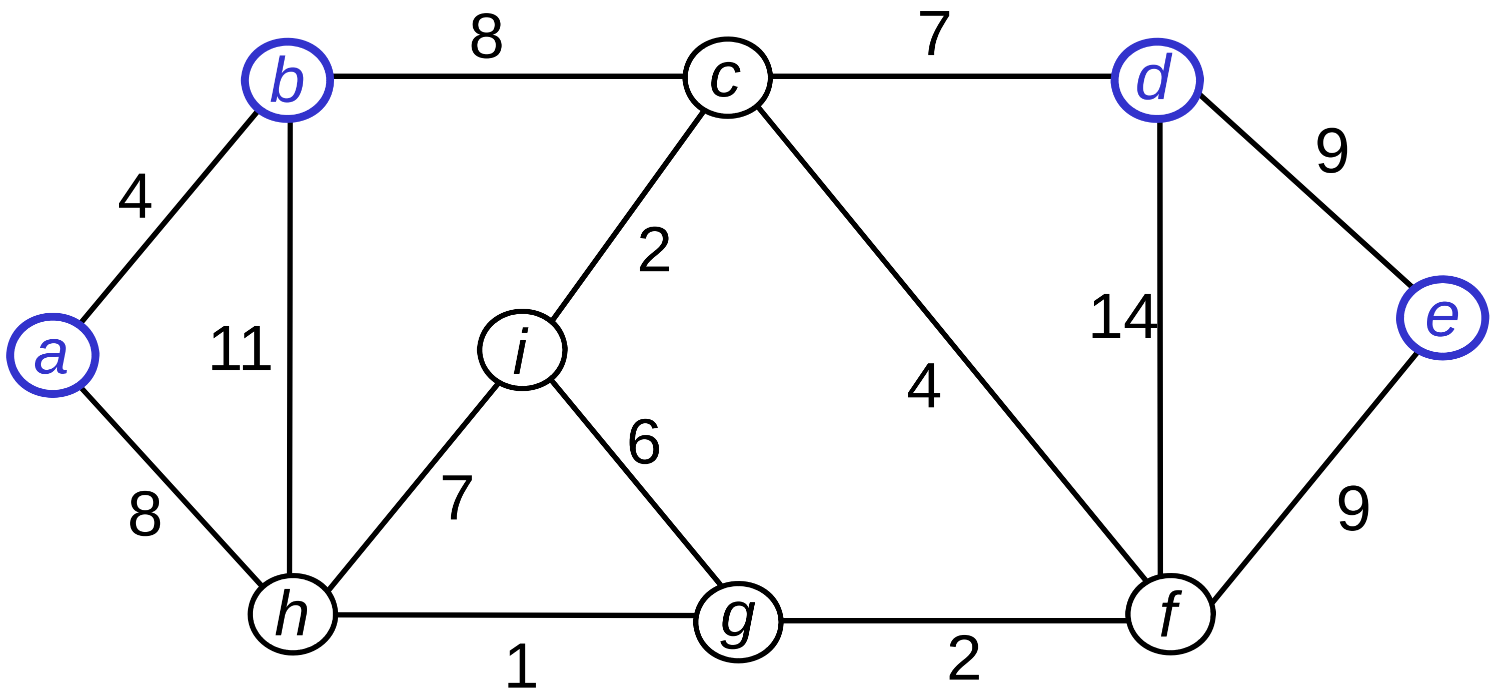
Cuts

A cut respects a set *A* of edges if no edge in *A* crosses the cut.



Cuts

An edge is a light edge crossing a certain cut if its weight is the minimum of any edge crossing the cut.



3. Safe edges

Theorem: Let A be a subset of some minimum spanning tree T, and let $(S, V \setminus S)$ be a cut that respects A. If (u,v) is a light edge crossing $(S, V \setminus S)$ then (u,v) is safe for A.

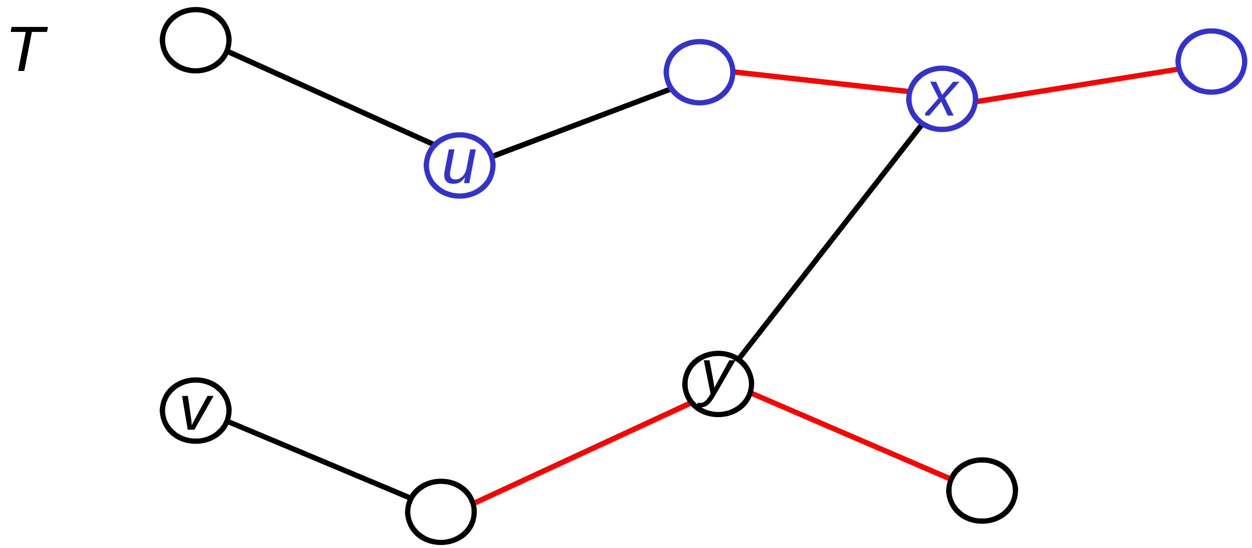
Proof:

Case 1: $(u,v) \in T$: ok

Case 2: $(u,v) \notin T$:

We construct another minimum spanning tree T' with $(u,v) \in T'$ and $A \subseteq T'$.

Safe edges



Adding (u,v) to T yields a cycle.

On this cycle, there is at least one edge (x,y) in T that also crosses the cut.

Safe edges

$$T' = T \setminus \{(x,y)\} \cup \{(u,v)\}$$

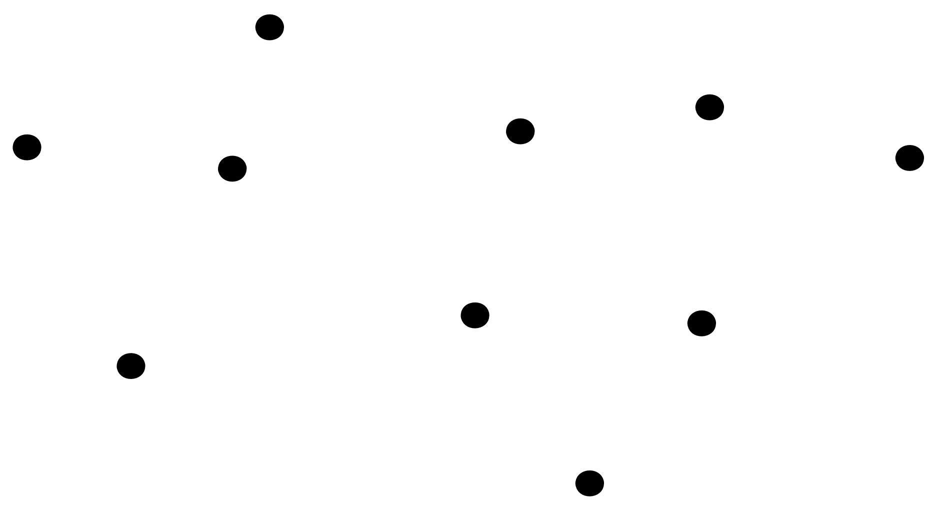
is a minimum spanning tree, since

$$w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$$

4. The graph G_A

$G_A = (V, A)$

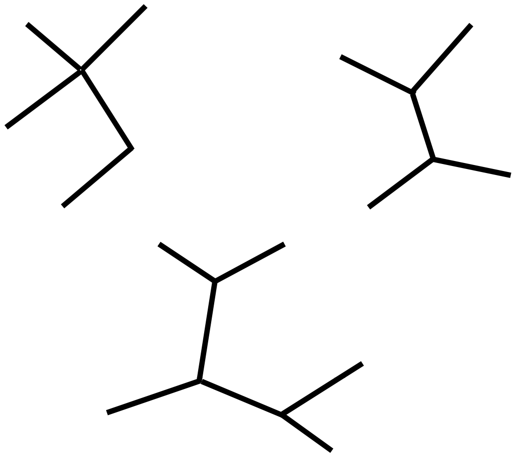
- is a forest, i.e. a collection of trees
- at the beginning, when $A = \emptyset$, each tree consists of a single vertex
- any safe edge for A connects distinct trees



The graph G_A

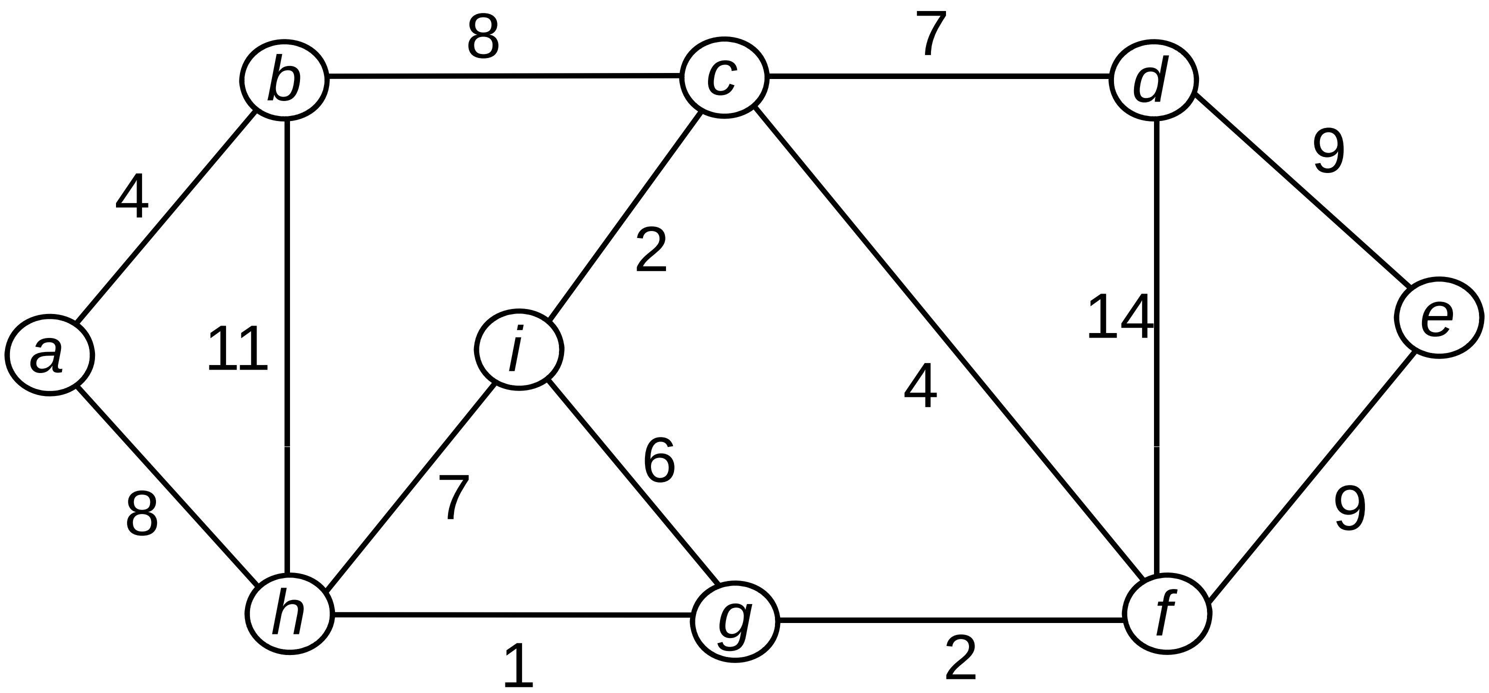
Corollary: Let B be a tree in $G_A = (V, A)$. If (u,v) is a light edge connecting B to some other tree in G_A, then (u,v) is safe for A.

Proof: $(B, V \setminus B)$ respects A and (u,v) is a light edge for this cut.



5. Kruskal's algorithm

Always choose an edge of smallest weight that connects two trees B_1 and B_2 in G_A.



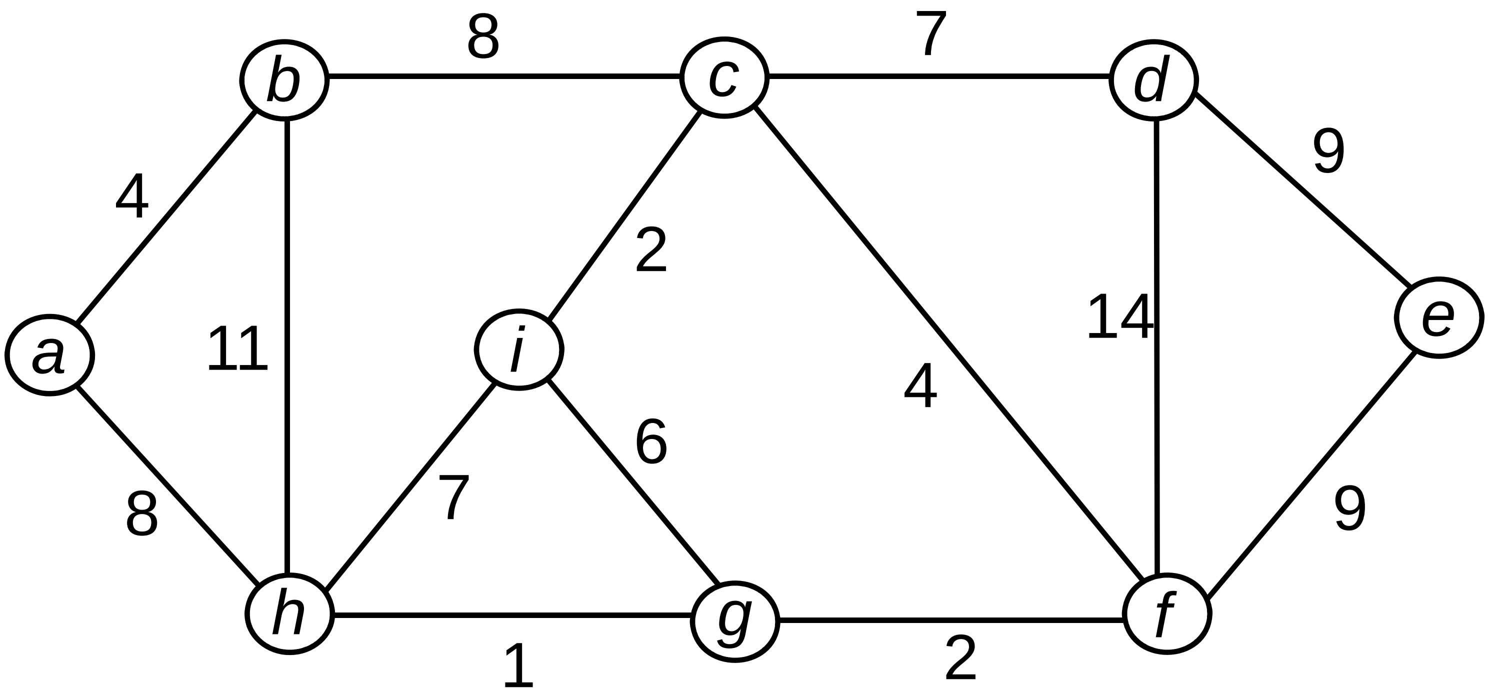
Kruskal's algorithm

1. $A \leftarrow \emptyset$;
2. **for all** $v \in V$ **do** $B_v \leftarrow \{ v \}$; **endfor;**
3. Generate a list L of all edges in E, sorted in non-decreasing order of weight;
4. **for all** (u,v) in L **do**
5. $\quad B_1 \leftarrow$ FIND(u); $B_2 \leftarrow$ FIND(v);
6. $\quad$ **if** $B_1 \neq B_2$ **then**
7. $\quad\quad A \leftarrow A \cup \{(u,v)\}$; UNION (B_1, B_2);
8. $\quad$ **endif;**
9. **endfor;**

Running time: $O(m\, \alpha(m,n) + m + n \log n)$

6. Prim's algorithm

A is always a single tree. Start from an arbitrary root vertex *r*. In each step, add a light edge to *A* that connects *A* to a vertex in *V* \ *A*.



Implementation

Q : priority queue containing all vertices $v \in V \setminus A$.

key of vertex v : minimum weight of any edge connecting v to a vertex in A (i.e. in the tree)

For a vertex v, let $p[v]$ denote the parent of v in the tree.

$A = \{ (v, p[v]) : v \in V - \{r\} - Q\}$

Prim's algorithm

```
for all v ∈ V do Insert(Q, ∞, v); endfor;
Choose a root vertex r ∈ V;
DecreaseKey(Q, 0, r);  p[r] ← nil;
while ¬Empty(Q) do
      (d, u) ← DeleteMin(Q);
      for all (u,v) ∈ E do
          if v ∈ Q and w(u,v) < key of v then
              DecreaseKey(Q, w(u,v), v);   p[v] ← u;
          endif;
      endfor;
endwhile;
```

Running time: $O(n \log n + m)$