Algorithms Theory

13 – Bin Packing

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Bin packing

1. Problem definition and general observations

2. Approximation algorithms for the online bin packing problem

3. Approximation algorithms for the offline bin packing problem

Problem definition

Bin Packing

Given:

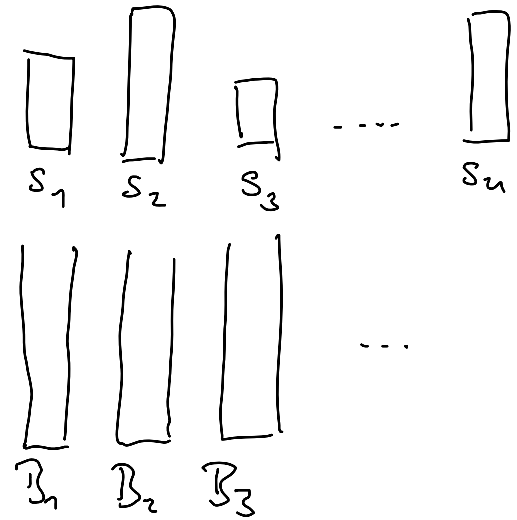
n items with sizes

$s_1, \ldots, s_n$

where $0 < s_i \leq 1$ for $1 \leq i \leq n$.

Items:

Bins:

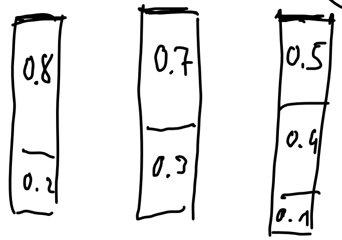


Goal:

Pack items into a minimum number of unit-capacity bins.

Example:

7 items with sizes 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8



Problem definition

Online bin packing:

Items arrive one by one. Each item must be assigned immediately to a bin, without knowledge of any future items. Reassignment is not allowed.

Offline bin packing:

All n items are known in advance, i.e. before they have to be packed.

Observations

- Bin packing is provably hard.
  (Offline bin packing is NP-hard.
  Decision problem is NP-complete.)

- There exists no online bin packing algorithm that always finds an optimal solution.

Online bin packing

Theorem 1:

There are inputs that force each online bin packing algorithm to use at least *4/3 OPT* bins where *OPT* is the minimum number of bins possible.

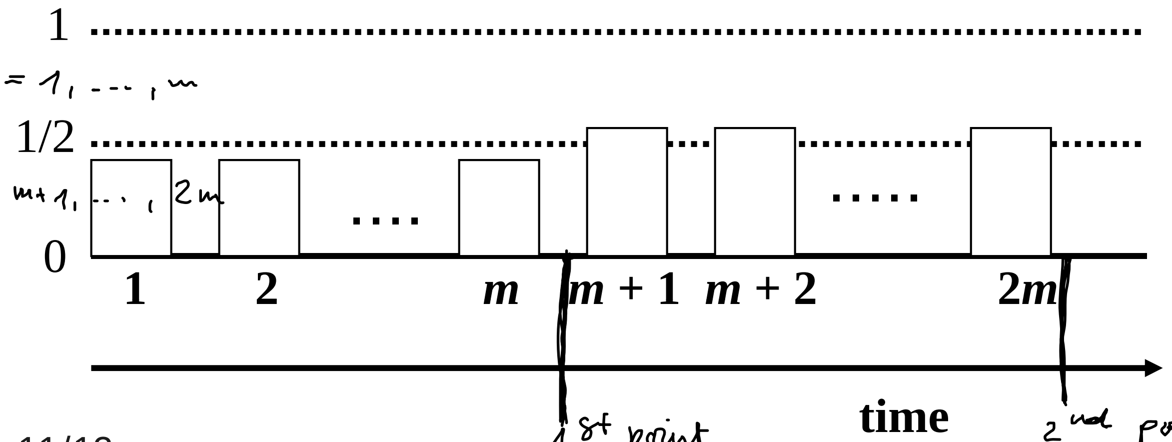
Proof: By contradiction

Assumption: online bin packing algorithm *A* always uses less than *4/3 OPT* bins

instance

$s_i = \frac{1}{2} - \varepsilon \quad i = 1, \dots, m$

$s_i = \frac{1}{2} + \varepsilon \quad i = m+1, \dots, 2m$



1st point

2nd point

Online bin packing

$\frac{1}{2}-\varepsilon$
$\frac{1}{2}-\varepsilon$

1st point of time:

$OPT = m/2$ and #bins(A) = b
by assumption: $b < 4/3 \cdot m/2 = 2/3\ m$

$b < 2/3 \cdot m$

Let $b = b_1 + b_2$, with
b_1 = #bins containing one item
b_2 = #bins containing two items

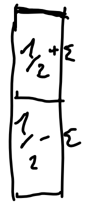
$m = b_1 + 2 \cdot b_2$

There is: $b_1 + 2\ b_2 = m$, i.e. $b_1 = m - 2b_2$

$b_1 = m - 2\ b_2$

Hence: $b = b_1 + b_2 = m - b_2$ (*)

Online bin packing



2nd point of time:

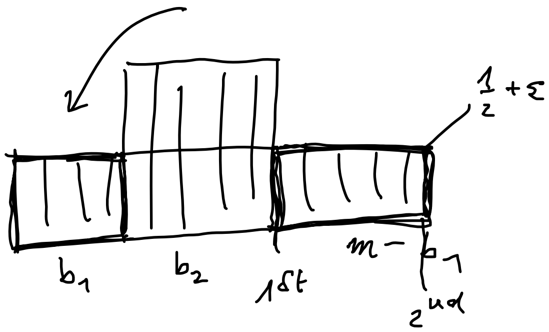
$b = b_1 + b_2$

$OPT = m$

#bins(A) $\geq \underbrace{b} + \underbrace{m - b_1} = m + b_2$

Assumption: $m + b_2 \leq$ #bins(A) $<$ 4/3m $\quad 1 - m$

$\Rightarrow \quad b_2 \leq m/3$



$\Longrightarrow$ using (∗): $b = m - b_2 >$ 2/3m

$b = m - b_2$

↯

Online bin packing

Next Fit (NF), First Fit (FF), Best Fit (BF)

Next Fit:

Assign an arriving item to the same bin as the preceding item. If it does not fit, open a new bin and place it there.

Theorem 2:

(a) For all input sequences I :

$$NF(I) \leq 2 \cdot OPT(I).$$

(b) There exist input sequences I such that:

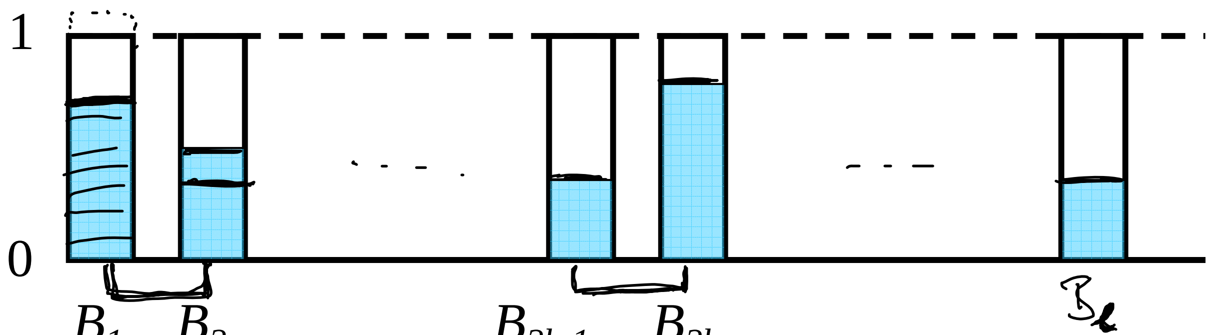
$$NF(I) \geq 2 \cdot OPT(I) - 2.$$

Next Fit

Proof: (a)

Consider two bins B_{2k-1}, B_{2k}, $2k \le NF(I)$.

Solution of NF



$s(B_1) + s(B_2) > 1$ $s(B_{2k-1}) + s(B_{2k}) > 1$

$$OPT(I) = \left\lceil \sum_{i=1}^{m} s_i \right\rceil$$

$$= \left\lceil \sum_{k=1}^{\lfloor \ell/2 \rfloor} s(B_{2k-1}) + s(B_{2k}) + s(B_\ell) \right\rceil$$

$$\ge \left\lceil \sum_{k=1}^{\lfloor \ell/2 \rfloor} 1 + s(B_\ell) \right\rceil$$

$$\ge \ell/2 = NF(I)/2 \quad | \cdot 2$$

$$\Rightarrow NF(I) \le 2 \cdot OPT(I) \quad \boxtimes$$

Next Fit

Proof: (b)

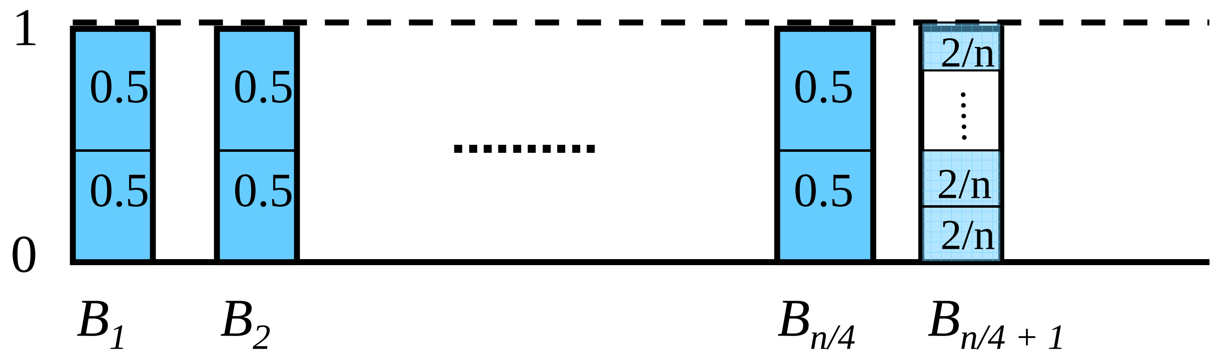
Consider an input sequence I of length n
($n \equiv 0 \pmod 4$):

$\frac{n}{2} \times \frac{1}{2}$

$\frac{n}{2} \times \frac{2}{n}$

0.5, $2/n$, 0.5, $2/n$, 0.5, ... , 0.5, $2/n$

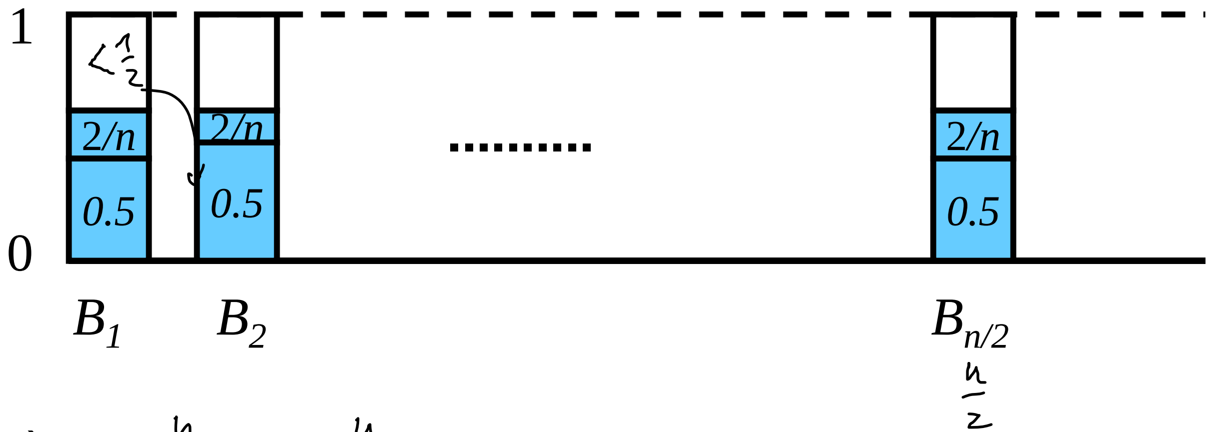
Optimal packing:



$\frac{n}{4} + 1$

Next Fit

Next Fit yields:



$NF(I) = \frac{n}{2} = \frac{n}{2} + 2 - 2 = 2 \cdot OPT(I) - 2$

$OPT(I) = \frac{n}{4} + 1$