Algorithms Theory

14 – Dynamic Programming (2)

Matrix-chain multiplication

P.D. Dr. Alexander Souza

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Optimal substructure

Dynamic programming is typically applied to
optimization problems.

An optimal solution to the original problem contains
optimal solutions to smaller subproblems.

Matrix-chain multiplication

Given: sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \ldots . \cdot A_n$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is either a single matrix or the product of two fully parenthesized matrix products, surrounded by parentheses.

Examples of fully parenthesized matrix products of the chain $\langle A_1, A_2, \ldots, A_n \rangle$

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$ are:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

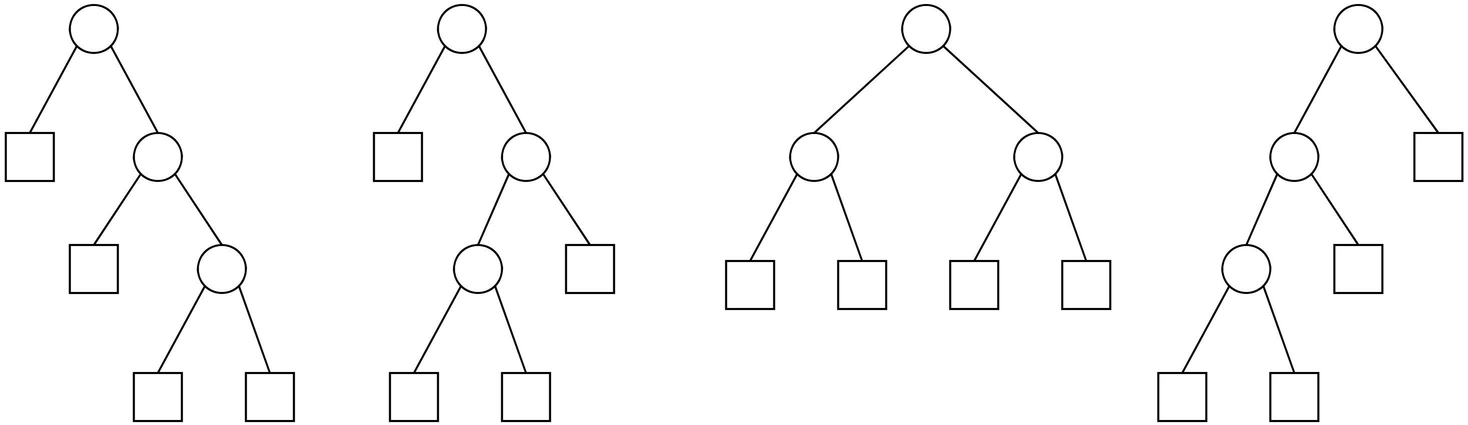
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

$$(((A_1A_2)A_3)A_4)$$

Number of different parenthesizations

Different parenthesizations correspond to different trees:



Number of different parenthesizations

Let *P(n)* be the number of alternative parenthesizations of the product $A_1 \ldots A_k A_{k+1} \ldots A_n$.

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) P(n-k) \quad \text{for } n \geq 2$$

$$P(n+1) = \frac{1}{n+1} \binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \quad n-\text{th Catalan number}$$

Remark: Determining the optimal parenthesization by exhaustive search is not reasonable.

Multiplying two matrices

$$A = (a_{ij})_{p \times q}, B = (b_{ij})_{q \times r}, A \cdot B = C = (c_{ij})_{p \times r},$$

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}.$$

```
Algorithm Matrix-Mult
Input:    (p × q) matrix A, (q × r) matrix B
Output:   (p × r) matrix C = A · B
for i := 1 to p do
    for j :=1 to r do
        C[i, j] := 0
        for k := 1 to q do
            C[i, j] := C[i, j] + A[i, k] ·B[k,j]
```

Number of multiplications and additions: $p \cdot q \cdot r$

Remark: Using this algorithm, multiplying two ($n \times n$) matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (10×100) matrix

A_2 : (100×5) matrix

A_3 : (5×50) matrix

a) Parenthesization $((A_1 A_2) A_3)$ requires

$A' = (A_1 A_2)$:

$A' A_3$:

Sum:

Matrix-chain multiplication: Example

A_1 : (10×100) matrix

A_2 : (100×5) matrix

A_3 : (5×50) matrix

a) Parenthesization $(A_1 (A_2 A_3))$ requires

$A'' = (A_2 A_3)$:

$A_1 A''$:

Sum:

Structure of an optimal parenthesization

$(A_{i...j}) = ((A_{i...k})\ (A_{k+1....j}))$ $\quad i \le k < j$

Any optimal solution to the matrix-chain multiplication problem contains optimal solutions to subproblems.

Determining an optimal solution recursively:

Let m[i,j] be the minimum number of operations needed to compute the product $A_{i...j}$:

m[i,j] = 0, if $i = j$

$$m[i,j] = \min_{i \le k < j} \{ m[i,k] + m[k+1, j] + p_{i-1} p_k p_j \}, \text{ otherwise}$$

s[i,j] = optimal splitting value *k*, i.e. the optimal parenthesization of $(A_{i...j})$ splits the product between A_k and A_{k+1}

Recursive matrix-chain multiplication

Algorithm *rec-mat-chain*(p, i, j)

Input: sequence $p = \langle p_0, p_1, \ldots, p_n \rangle$,

where $(p_{i-1} \times p_i)$ is the dimensionen of matrix A_i

Invariant: rec-mat-chain(p, i, j) returns $m[i, j]$

```
if i = j then return 0
m[i, j] := ∞
for k := i to j – 1 do
      m[i, j] := min( m[i,j],   p_{i-1} p_k p_j +
                                rec-mat-chain(p, i, k) +
                                rec-mat-chain(p, k+1, j) )
return m[i, j]
```

Initial call: rec-mat-chain(p,1, n)

Recursive matrix-chain multiplication: Running time

Let $T(n)$ be the time taken by rec-mat-chain(p,1,n).

$$T(1) \geq 1$$

$$T(n) \geq 1 + \sum_{k=1}^{n-1} \left(T(k) + T(n-k) + 1\right)$$

$$\geq n + 2\sum_{i=1}^{n-1} T(i)$$

$$\Rightarrow T(n) \geq 3^{n-1} \quad \text{(induction)}$$

Exponential running time!

Matrix-chain multiplication dynamic programming

Algorithm *dyn-mat-chain*

Input: sequence $p = \langle p_0, p_1, \ldots, p_n \rangle$, $(p_{i-1} \times p_i)$ the dimension of matrix A_i

Output: $m[1,n]$

$n := \text{length}(p) - 1$
for $i := 1$ **to** n **do** $m[i, i] := 0$
for $l := 2$ **to** n **do** /* l = length of the subproblem */
  for $i := 1$ **to** $n - l + 1$ **do** /* i is the left index */
    $j := i + l - 1$ /* j is the right index */
    $m[i, j] := \infty$
    for $k := i$ **to** $j - 1$ **do**
      $m[i, j] := \min(\, m[i, j],\; p_{i-1}\, p_k\, p_j + m[i, k] + m[k + 1, j] \,)$
return $m[1, n]$

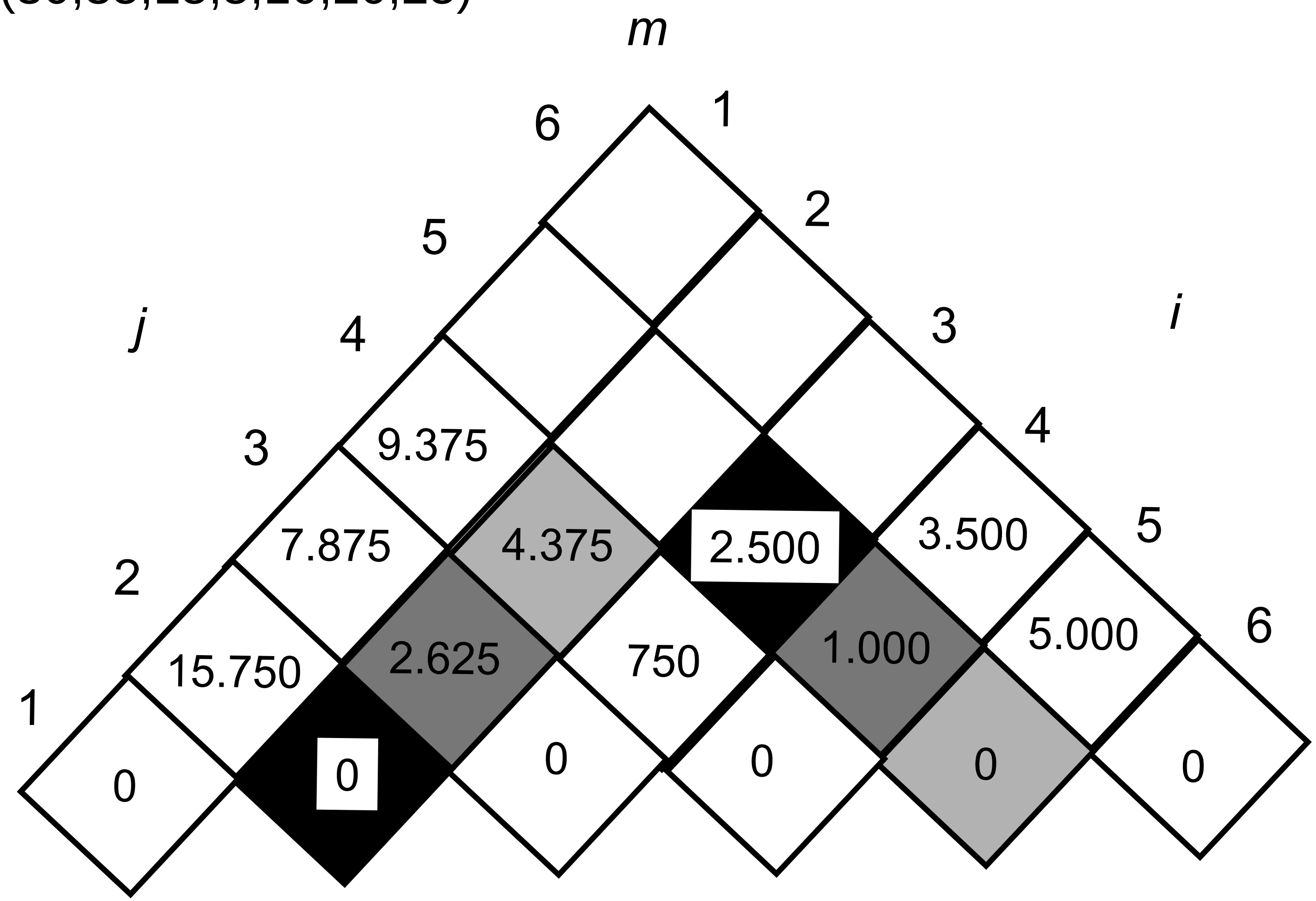
Example

A_1	(30×35)	A_4	(5×10)
A_2	(35×15)	A_5	(10×20)
A_3	(15×5)	A_6	(20×25)

$p = (30,35,15,5,10,20,25)$

Example

$p = (30,35,15,5,10,20,25)$



Example

$$m[2,5] =$$

$$\min_{2 \le k < 5} \left(m[2,k] + m[k+1,5] + p_1 p_k p_5 \right)$$

$$\min \begin{cases} m[2,2] + m[3,5] + p_1 p_2 p_5 \\ m[2,3] + m[4,5] + p_1 p_3 p_5 \\ m[2,4] + m[5,5] + p_1 p_4 p_5 \end{cases}$$

$$\min \begin{cases} 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000 \\ 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125 \\ 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375 \end{cases}$$

$$= 7125$$

Matrix-chain multiplication and optimal splitting values using dynamic programming

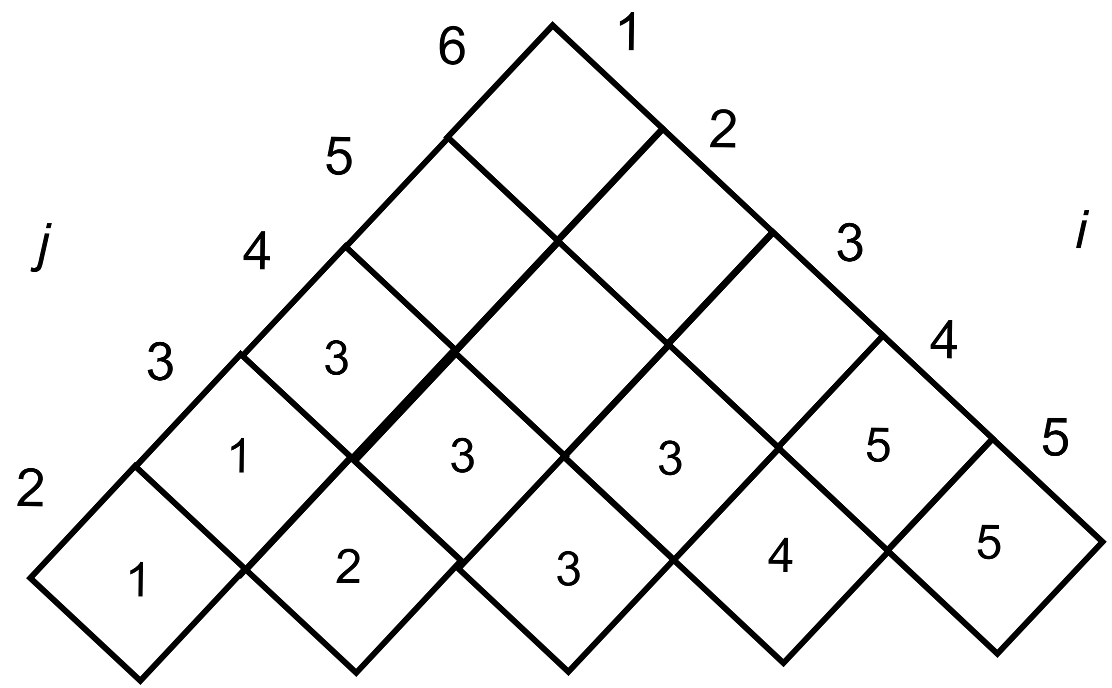
Algorithm *dyn-mat-chain(p)*

Input: sequence $p = \langle p_0, p_1, \ldots, p_n \rangle$, $(p_{i-1} \times p_i)$ the dimension of matrix A_i

Output: $m[1,n]$ and a matrix $s[i,j]$ containing the optimal splitting values

```
n := length(p) – 1
for i := 1 to n do m[i, i] := 0
for l := 2 to n do
    for i := 1 to n – l +1 do
        j := i + l – 1
        m[i, j] := ∞
        for k := i to j – 1 do
            q := m[i, j]
            m[i, j] := min( m[i, j],  p_{i-1} p_k p_j + m[i, k] + m[k + 1, j] )
            if m[i, j] < q then s[i, j] := k
return (m[1, n], s)
```

Example of splitting values



Computation of an optimal parenthesization

Algorithm *Opt-Parens*

Input: chain A of matrices, matrix s containing the optimal splitting values, two indices i and j

Output: an optimal parenthesization of $A_{i...j}$

if $i < j$
    then $X := \textit{Opt-Parens}(A, s, i, s[i, j])$
          $Y := \textit{Opt-Parens}(A, s, s[i, j] + 1, j)$
          return $(X \cdot Y)$
    else return A_i

Initial call: $\textit{Opt–Parens}(A, s, 1, n)$

Matrix-chain multiplication using dynamic programming (top-down approach)

„Memoization“ for increasing the efficiency of a recursive solution:

Only the *first time* a subproblem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned (without repeated computation!).

Memoized matrix-chain multiplication („notepad method“)

Algorithm *mem-mat-chain*(p, i, j)

Invariant: mem-mat-chain(p, i, j) returns $m[i, j]$;
the value is correct if $m[i, j] < \infty$

if $i = j$ **then return** 0
if $m[i, j] < \infty$ **then return** $m[i, j]$
for $k := i$ **to** $j - 1$ **do**
    $m[i, j]$:= min($m[i, j]$, $p_{i-1}\, p_k\, p_j$ +
        mem-mat-chain(p, i, k) +
        mem-mat-chain($p, k + 1, j$))
return $m[i, j]$

Memoized matrix-chain multiplication

Call:

```
n := length(p) – 1
for i := 1 to n do
    for j := 1 to n do
        m[i, j] := ∞
mem-mat-ket(p,1,n)
```

The computation of all entries $m[i, j]$ using *mem-mat-chain* takes $O(n^3)$ time.

$O(n^2)$ entries
each entry $m[i, j]$ is computed once
each entry $m[i, j]$ is looked up during the computation of $m[i', j']$ if $i' = i$ and $j' > j$ or $j' = j$ and $i' < i$
→ $m[i, j]$ is looked up during the computation of at most $2n$ entries

Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time $O(n \log n)$.

2. There is a linear time algorithm that determines a parenthesization using at most $1.155 \cdot M_{opt}$ multiplications.