Lower Bounds for Maximal Matchings and Maximal Independent Sets

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Two classical graph problems

Maximal matching

Maximal independent set
Two classical graph problems

- Maximal matching
- Maximal independent set

- Very easy to solve in the centralized setting: greedily add edges/nodes until not possible
Two classical graph problems

- **Maximal matching**
  - Very easy to solve in the *centralized* setting: greedily add edges/nodes until not possible

- **Maximal independent set**

- Can these problems be *solved efficiently* in a *distributed* setting?
Distributed setting (LOCAL model)

- **Graph** = communication network
- **Synchronous** rounds
- Time complexity = number of rounds required to solve the problem
- Nodes have IDs
Simple scenario

- Nodes are 2 colored
- The communication graph is $\Delta$-regular
computer network with port numbering

bipartite, 2-colored graph

Δ-regular (here Δ = 3)

output: maximal matching
Very simple algorithm

unmatched white nodes:
send *proposal* to port 1
**Very simple algorithm**

**unmatched white nodes:**
send *proposal* to port 1

**black nodes:**
*accept* the first proposal you get, *reject* everything else
(break ties with port numbers)
Very simple algorithm

unmatched white nodes:
send *proposal* to port 1

black nodes:
*accept* the first proposal you get, *reject* everything else
(break ties with port numbers)
Very simple algorithm

unmatched white nodes: send *proposal* to port 2
Very simple algorithm

unmatched white nodes: send *proposal* to port 2

black nodes: *accept* the first proposal you get, *reject* everything else (break ties with port numbers)
Very simple algorithm

unmatched white nodes:
send *proposal* to port 2

black nodes:
*accept* the first proposal you get, *reject* everything else
(break ties with port numbers)
Very simple algorithm

unmatched white nodes:
send *proposal* to port 3
Very simple algorithm

unmatched white nodes: send *proposal* to port 3

black nodes: *accept* the first proposal you get, *reject* everything else (break ties with port numbers)
**Very simple algorithm**

unmatched white nodes: send *proposal* to port 3

black nodes: *accept* the first proposal you get, *reject* everything else (break ties with port numbers)
Very simple algorithm

Finds a *maximal matching* in $O(\Delta)$ communication rounds
Very simple algorithm

Finds a maximal matching in $O(\Delta)$ communication rounds

This is optimal!
Related work

Maximal matching, LOCAL model, \(O(f(\Delta) + g(n))\)

**Algorithms:**
- deterministic
- randomized

**Lower bounds:**
- deterministic
- randomized
Maximal matching, LOCAL model, \( O(f(\Delta) + g(n)) \)

**Algorithms:**
- \( \bigcirc \) deterministic
- \( \bullet \) randomized

**Lower bounds:**
- \( \square \) deterministic
- \( \blush \) randomized

polylog(n) deterministic

Maximal matching, LOCAL model, $O(f(\Delta) + g(n))$

Algorithms:
- deterministic
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Lower bounds:
- deterministic
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- Hanckowiak et al. (1998)
- Hanckowiak et al. (2001)
- Fischer (2017)
- Israeli & Itai (1986)
Maximal matching, LOCAL model, \(O(f(\Delta) + g(n))\)

**Algorithms:**
- \(\circ\) deterministic
- \(\bullet\) randomized

**Lower bounds:**
- \(\square\) deterministic
- \(\blacksquare\) randomized

- \(O(\Delta + \log^* n)\) deterministic
- \(\log^* n\)
- \(\log n\)
- \(\log^3 n\)
- \(\log^4 n\)
- \(\log^7 n\)
- \(\log\log\log n\)
- \(\log\log\log\log n\)
- \(\log\log n\)
- \(\log n\)
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- Panconesi & Rizzi (2001)

Algorithms:
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Maximal matching, LOCAL model, $O(f(\Delta) + g(n))$

**Algorithms:**
- ○ deterministic
- ● randomized

**Lower bounds:**
- □ deterministic
- ◼ randomized

- Kuhn et al. (2004, 2016)
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**Formulas:**
- $\log^7 n$ to $\log^4 n$ to $\log^3 n$ to $\log n$
- $\sqrt{\frac{\log n}{\log \log n}}$
- $\log \Delta$ to $\frac{\log \Delta}{\log \log \Delta}$
Maximal matching, LOCAL model, $O(f(\Delta) + g(n))$

**Algorithms:**
- Deterministic
- Randomized

**Lower bounds:**
- Deterministic
- Randomized

**Kuhn et al. (2004, 2016)**

$O(\log \Delta + \log^* n)$ ???
Maximal matching, LOCAL model, $O(f(\Delta) + g(n))$

**Algorithms:**
- ○ deterministic
- ● randomized

**Lower bounds:**
- □ deterministic
- ■ randomized

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Kuhn et al. (2004, 2016)

Barenboim et al. (2012, 2016)

Fischer (2017)

Hanckowiak et al. (1998)

Hanckowiak et al. (2001)

Israeli & Itai (1986)

Panconesi & Rizzi (2001)
Maximal matching,\nLOCAL model,\n$O(f(\Delta) + g(n))$\n
Algorithms:\n○ deterministic\n● randomized\n
Lower bounds:\n□ deterministic\n□ randomized

Main results

Maximal matching and maximal independent set cannot be solved in

- $o(\Delta + \log \log n / \log \log \log n)$ rounds with randomized algorithms
- $o(\Delta + \log n / \log \log n)$ rounds with deterministic algorithms

Upper bound: $O(\Delta + \log^* n)$
Proof sketch

Maximal matching in $o(\Delta)$ rounds

What we really care about
Proof sketch

Maximal matching in $o(\Delta)$ rounds
→ “$\Delta^{1/2}$ matching” in $o(\Delta^{1/2})$ rounds

k-matching: select at most $k$ edges per node
Proof sketch

Maximal matching in $o(\Delta)$ rounds
→ “$\Delta^{1/2}$ matching” in $o(\Delta^{1/2})$ rounds
→ $P(\Delta^{1/2}, 0)$ in $o(\Delta^{1/2})$ rounds

$P(x,y)$: unnatural relaxed variant of maximal matching
Proof sketch

Maximal matching in $o(\Delta)$ rounds
→ “$\Delta^{1/2}$ matching” in $o(\Delta^{1/2})$ rounds
→ $P(\Delta^{1/2}, 0)$ in $o(\Delta^{1/2})$ rounds
→ $P(O(\Delta^{1/2}), o(\Delta))$ in 0 rounds

$P(x,y)$: unnatural relaxed variant of maximal matching
Proof sketch

Maximal matching in $o(\Delta)$ rounds
→ “$\Delta^{1/2}$ matching” in $o(\Delta^{1/2})$ rounds
→ $P(\Delta^{1/2}, 0)$ in $o(\Delta^{1/2})$ rounds
→ $P(O(\Delta^{1/2}), o(\Delta))$ in 0 rounds
→ contradiction

$P(x,y)$: unnatural relaxed variant of maximal matching
Round elimination technique

• Given:
  • algorithm $A_0$ solves problem $P_0$ in $T$ rounds

• We construct:
  • algorithm $A_1$ solves problem $P_1$ in $T - 1$ rounds
  • algorithm $A_2$ solves problem $P_2$ in $T - 2$ rounds
  • algorithm $A_3$ solves problem $P_3$ in $T - 3$ rounds
  ... 
  • algorithm $A_T$ solves problem $P_T$ in 0 rounds

• But $P_T$ is nontrivial, so $A_0$ cannot exist
Round elimination technique

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    ...
  • algorithm $A_T$ solves problem $P_T$ in 0 rounds

• But $P_T$ is nontrivial, so $A_0$ cannot exist

[B. 2019]: Given any $P_i$, it is possible to find $P_{i+1}$ automatically, but the description of the problem may grow exponentially
Proof sketch

Maximal matching in $o(\Delta)$ rounds
→ "$\Delta^{1/2}$ matching" in $o(\Delta^{1/2})$ rounds
→ $P(\Delta^{1/2}, 0)$ in $o(\Delta^{1/2})$ rounds
→ $P(O(\Delta^{1/2}), o(\Delta))$ in 0 rounds
→ contradiction

Apply round elimination technique
Main Lemma

• Given: $A$ solves $P(x, y)$ in $T$ rounds

• We can construct: $A'$ solves $P(x+1, y+x)$ in $T-1$ rounds

\[
W_\Delta(x, y) = \left( \text{MO}^{d-1} \mid P^d \right) O^y X^x,
\]
\[
B_\Delta(x, y) = \left( [MX][POX]^{d-1} \mid [OX]^d \right) [POX]^y [MPOX]^x,
\]
\[
d = \Delta - x - y
\]
Lower bound for the LOCAL model

- The lower bound holds for the *simple scenario* where randomness is not allowed and nodes are anonymous.

- **Additional steps** are required to handle:
  - randomness
  - non anonymous nodes
Conclusions and open problems

- *Linear-in-Δ lower bounds* for maximal matchings and maximal independent sets

- Maximal matchings can not be solved fast:
  - The simple proposal algorithm is optimal
  - Randomization and large messages do not help

- How about a lower bound for distributed $\Delta+1$ coloring?