New Classes of Distributed Time Complexity

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Based on

- **New Classes of Distributed Time Complexity** - Alkida Balliu, Juho Hirvonen, Janne H. Korhonen, Tuomo Lempiäinen, Dennis Olivetti, and Jukka Suomela [STOC’18]
- **Almost Global Problems in the LOCAL Model** - Alkida Balliu, Sebastian Brandt, Dennis Olivetti, and Jukka Suomela [Submitted]

Slides based on “New Classes of Distributed Time Complexity”, Janne H. Korhonen
Outline

- LOCAL Model
- Locally Checkable Labellings
- Results
- Proof idea
LOCAL Model

- Distributed
- Unlimited bandwidth
- Unlimited computational power
- Nodes have IDs
- In this talk:
  - deterministic algorithms
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Locally Checkable Labellings

LCL Problems:

- Introduced by Naor and Stockmeyer in 1995
- Constant-size input labels
- Constant-size output labels
- The maximum degree is constant
- Validity of the output is locally checkable
There are only three possible time complexities:

- $O(1)$: trivial problems
- $O(\log^*n)$: local problems (symmetry breaking)
- $O(n)$: global problems
LCL on Cycles

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  - $\Theta(1)$: trivial problems
  - $\Theta(\log^* n)$: local problems (symmetry breaking)
  - $\Theta(n)$: global problems

- Automatic speedups:
  - Any $o(\log^* n)$-rounds algorithm can be converted to a $O(1)$-rounds algorithm [Naor and Stockmeyer, 1995]
  - Any $o(n)$-rounds algorithm can be converted to a $O(\log^* n)$-rounds algorithm [Chang, Kopelowitz and Pettie, 2016]
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- On cycles with no input, given an LCL description, we can decide its time complexity. [Naor and Stockmeyer, 1995] [Brandt et al, 2017]
LCL on Cycles

1

log* n

n
LCL on Trees

[Chang and Pettie, 2017]:

- Any $n^{o(1)}$-rounds algorithm can be converted to a $O(\log n)$-rounds algorithm
- There are problems of complexity $\Theta(n^{1/k})$
LCL on Trees

1 \log \log^* n \log^* n \log n \quad n^{\alpha(1)} \quad \ldots \quad n^{1/4} \quad n^{1/3} \quad n^{1/2} \quad n
LCL on Trees (Our Results)

\[
\begin{align*}
1 & \quad \log \log^* n & \quad \log^* n & \quad \log n & \quad n^{\alpha(1)} & \quad \ldots & \quad n^{1/4} & \quad n^{1/3} & \quad n^{1/2} & \quad n \\
\bullet & \quad ? & \quad \bullet & \quad ? & \quad \bullet & \quad ? & \quad \bullet & \quad ? & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
\]
LCL on General Graphs

- There are problems with complexity $\Theta(\log n)$ [Brandt et al, 2016] [Chang, Kopelowitz and Pettie, 2016] [Ghaffari and Su, 2017]
LCL on General Graphs

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- Many problems require $\Omega(\log n)$ and $O(\text{poly log } n)$
- Different scenario with randomized algorithms
LCL on General Graphs

1 \log \log^* n \log^* n \log n \quad n^{(1)} \ldots n^{1/4} n^{1/3} n^{1/2} n
LCL on General Graphs?

\[ 1, \log \log^* n, \log^* n, \log n, n^{\alpha(1)}, \ldots, n^{1/4}, n^{1/3}, n^{1/2}, n \]
LCL on General Graphs (Our Results)

1. \log \log^* n
2. \log^* n
3. \log n
4. \ldots
5. n^{\alpha(1)}
6. n^{1/4}
7. n^{1/3}
8. n^{1/2}
9. n

LCL on General Graphs (Our Results)

\[
\begin{array}{cccccccc}
1 & \log \log n & \log^* n & \log n & n^{\omega(1)} & \ldots & n^{1/4} & n^{1/3} & n^{1/2} & n
\end{array}
\]
LCL on General Graphs (Our Results)

New Classes of Distributed Time Complexity
General Idea

- We start from an LCL problem $\Pi$ on cycles:
  - $\Pi_l$ has complexity $T(n) = \Theta(\log^* n)$
    - 3 colouring
  - $\Pi_g$ has complexity $T(n) = \Theta(n)$
    - a variant of 2 colouring

- We build a speed-up construction:
  - in $\ell$ rounds a node “sees” at distance $f(\ell) = \ell g(\ell)$
  - we obtain an easier version of $\Pi$
  - new complexity:
    - $\Theta(f^{-1}(T(n)))$
Example

- We start from a cycle
We add shortcuts on top of the cycle, $g(\ell) = 2^\ell$
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Example

- \(\Pi\) can be solved in \(\Theta(f^{-1}(T(n)))\) rounds using the shortcuts
Example

Problem: this is not a valid LCL
Valid LCL

- An LCL problem must be defined on any graph, not just on some “relevant” instances
- What if the shortcuts are missing?
- What if a cycle is not present at all?
Fixing the details

Input:
- a graph
- a proof that the graph is a relevant instance
  - it must be locally checkable

Output:
- Solve $\Pi$, or
- Prove that there is an error in the input proof, or in the graph structure
  - it must be locally checkable
Local checkability of the input
Correct instance
Correct instance
Incorrect instance
Incorrect instance
On incorrect instances, it should be easy to prove that there is an error
On correct instances, it should be impossible, or hard, to prove that there is an error
Using different $g(\ell)$

New Classes of Distributed Time Complexity
Using different $g(\ell)$

Which shortcut constructions can be locally checked?
Link Machine Programs

- Constant number of registers
- Reset
  - $r_1 \leftarrow 1$
- Addition
  - $r_1 \leftarrow r_2 + r_3$
- If statements with equality comparison
  - if $r_1 = r_2$
  - if $r_1 \neq r_2$
- $g(\ell) =$ value of the maximum register after $\ell$ executions of the program
Link Machine Programs: Examples

- \( g(\ell) = 2^\ell \)
  - \( r_1 \leftarrow r_1 + r_1 \)
Link Machine Programs: Examples

- \( g(\ell) = 2^\ell \)
  - \( r_1 \leftarrow r_1 + r_1 \)

- \( g(\ell) = \Theta(\ell^2) \)
  - \( r_1 \leftarrow r_1 + 1 \)
  - \( r_2 \leftarrow r_2 + r_1 \)
## Link Machine Programs: Building Blocks

<table>
<thead>
<tr>
<th>Program $P$</th>
<th>Input</th>
<th>Output</th>
<th>Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>COUNT</td>
<td>–</td>
<td>$y = \ell$</td>
<td>$\ell$</td>
</tr>
<tr>
<td>$\text{ROOT}'_k$</td>
<td>–</td>
<td>$y = \Theta(\ell^{1/k})$</td>
<td>$\Theta(\ell^{1/k})$</td>
</tr>
<tr>
<td>$\text{ROOT}_k$</td>
<td>$x$</td>
<td>$y = \Theta(x^{1/k})$</td>
<td>$\Theta(x)$</td>
</tr>
<tr>
<td>$\text{POW}_k$</td>
<td>$x$</td>
<td>$y = \Theta(x^k)$</td>
<td>$\Theta(x^k)$</td>
</tr>
<tr>
<td>EXP</td>
<td>$x$</td>
<td>$y = 2^\Theta(x)$</td>
<td>$2^\Theta(x)$</td>
</tr>
<tr>
<td>LOG</td>
<td>$x$</td>
<td>$y = \Theta(\log x)$</td>
<td>$\Theta(x)$</td>
</tr>
</tbody>
</table>
### Link Machine Programs: $g(\ell)$

<table>
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<tr>
<th>Program $P$</th>
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<tbody>
<tr>
<td>$\text{POW}_p \circ \text{ROOT}'_q$</td>
<td>$\Theta(\ell p/q)$</td>
</tr>
<tr>
<td>$\text{EXP} \circ \text{POW}_q \circ \text{ROOT}'_p$</td>
<td>$(p \geq q)$ $2^{\Theta(\ell q/p)}$</td>
</tr>
<tr>
<td>$\text{EXP} \circ \text{POW}_q \circ \text{ROOT}_p \circ \text{LOG} \circ \text{COUNT}$</td>
<td>$(p \geq q)$ $2^{\Theta(\log q/p \ell)}$</td>
</tr>
</tbody>
</table>
Results

New Classes of Distributed Time Complexity

\[ \log \log^* n \]

\[ \log^* n \]

\[ \log \frac{p}{q} \log^* n \]

\[ 2^{\log \frac{q}{p} \log^* n} \]

\[ (\log^* n)^{\frac{q}{p}} \]

\[ \log n \]

\[ n^{\alpha(1)} \]

\[ n^{1/4} \]

\[ n^{1/3} \]

\[ n^{1/2} \]

\[ n \]
Conclusions and Open Problems

- Are there other gaps on trees?
- What happens between $\Omega(\log \log^* n)$ and $O(\log^* n)$ on trees?
- What about polynomial complexities with sub-diameter time/sub-linear volume?
- What are meaningful subclasses of LCL problems where there are gaps again?
Thank you!