Vertex Cover in Graphs with Locally Few Colors

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Abstract. In [13], Erdős et al. defined the local chromatic number of a graph as the minimum number of colors that must appear within distance 1 of a vertex. For any $\Delta \geq 2$, there are graphs with arbitrarily large chromatic number that can be colored so that (i) no vertex neighborhood contains more than Δ different colors (*bounded local colorability*), and (ii) adjacent vertices from two color classes induce a complete bipartite graph (*biclique coloring*).

We investigate the weighted vertex cover problem in graphs when a locally bounded coloring is given. This generalizes the vertex cover problem in bounded degree graphs to a class of graphs with arbitrarily large chromatic number. Assuming the Unique Game Conjecture, we provide a tight characterization. We prove that it is UGC-hard to improve the approximation ratio of $2 - 2/(\Delta + 1)$ if the given local coloring is not a biclique coloring. A matching upper bound is also provided. Vice versa, when properties (i) and (ii) hold, we present a randomized algorithm with approximation ratio of $2 - \Omega(1)\frac{\ln \ln \Delta}{\ln \Delta}$. This matches known inapproximability results for the special case of bounded degree graphs.

Moreover, we show that the obtained result finds a natural application in a classical scheduling problem, namely the precedence constrained single machine scheduling problem to minimize the total weighted completion time. In a series of recent papers it was established that this scheduling problem is a special case of the minimum weighted vertex cover in graphs $G_{\mathbf{P}}$ of incomparable pairs defined in the dimension theory of partial orders. We show that $G_{\mathbf{P}}$ satisfies properties (i) and (ii) where $\Delta - 1$ is the maximum number of predecessors (or successors) of each job.

Keywords: approximation, local coloring, scheduling, vertex cover

1 Introduction

Vertex Cover is one of the most studied problems in combinatorial optimization: Given a graph $G = (V, E)$ with weights w_i on the vertices, find a subset $V' \subseteq V$, minimizing the objective function $\sum_{i \in V'} w_i$, such that for each edge $\{u, v\} \in E$, at least one of u and v belongs to V'.

The related bibliography is vast and cannot be covered in this introductory note. We mention here that vertex cover cannot be approximated within a factor of 1.3606 [11], unless P=NP. Moreover, if the Unique Game Conjecture (UGC) [23] holds, Khot and Regev [24] show that vertex cover is hard to approximate within any constant factor better than 2. On the other side several simple 2-approximation algorithms are known (see e.g. [30, 20]). Hochbaum [20] uses the natural linear program (LP) relaxation and a threshold rounding approach to obtain better than 2 approximation algorithms when a k-coloring of the graph is given as input. An optimal solution to the LP assigns a non-negative real value to each vertex of the input graph G. It is well known that such a solution is half-integral [30]. Let S_1 be the vertices of the input graph which attain value 1 in this solution; and let S_2 be the vertices which attain value 1/2. The vertices S_1 together with a cover of the subgraph induced by S_2 are sufficient to cover the whole graph G. A k-coloring of the subgraph induced by S_2 gives an independent set of value at least $w(S_2)/k$, where $w(S_2)$ is the sum of the vertex weights in S_2. This yields a $(2-2/k)$-approximation for the minimum weighted vertex cover problem. For graphs with degree bounded by d, this directly leads to a $(2-2/d)$-approximation.

This basic approach has been considerably improved by Halperin [19]. The improvement is obtained by replacing the LP relaxation with a stronger semidefinite program (SDP) relaxation, and using a fundamental result by Karger et al. [22]. The algorithm in [19] achieves a performance ratio of $2-(1-o(1))\frac{2\ln\ln d}{\ln d}$, which improves the previously known [18] ratio of $2-\frac{\ln d+O(1)}{d}$. Under the UGC [23], Austrin, Khot and Safra [5] have recently proved that it is NP-hard to approximate vertex cover in bounded degree graphs to within a factor $2-(1+o(1))\frac{2\ln\ln d}{\ln d}$. This exactly matches the algorithmic result of Halperin [19] up to the $o(1)$ term. For general vertex cover, the currently best approximation ratio is due to Karakostas [21], achieving a performance of $2-\Theta(1/\sqrt{\ln n})$.

Brook's theorem states that except for complete graphs and odd cycles, graphs with maximum degree d can be colored with d colors. In this paper we consider the vertex cover problem in graphs with bounded local chromatic number, a generalization of the bounded degree case with arbitrarily large chromatic number.

For an undirected graph $G=(V,E)$ and a vertex $u\in V$, we use $N(u):=\{v\in V:\{u,v\}\in E\}$ to denote the set of neighbors of u. Given a graph G, a valid vertex coloring of G is a function $\varphi: V\to\mathbb{N}$ such that $\varphi(u)\neq\varphi(v)$ whenever $\{u,v\}\in E$. In [13], Erdős et. al introduced the notion of local colorings.

Definition 1 (Local Coloring). *Let k be a positive integer. A k-local coloring of a graph G is a valid vertex coloring φ such that for every vertex $u\in V$, $|\{\varphi(v):v\in N(u)\}|<k$.*

Note that the definition implies that in each closed neighborhood $\{u\}\cup N(u)$, the number of different colors is bounded by k. The *local chromatic number* $\psi(G)$ of a graph G is the minimum k such that G admits a k-local coloring [13]. Since any valid coloring with k colors also is a local k-coloring, clearly, $\psi(G)\leq\chi(G)$. Interestingly, it can also be shown that the local chromatic number is always at least as large as the fractional chromatic number, i.e., $\psi(G)\geq\chi_f(G)$ [26].

Given a valid vertex coloring φ and an integer i, let $C_i := \{v \in V : \varphi(v) = i\}$ be the set of vertices with color i. Further, for integers $i \neq j$, we use $N_j(C_i) := C_j \cap \bigcup_{v \in C_i} N(v)$ to denote the vertices with color j that have a neighbor with color i. We consider colorings with the following density condition.

Definition 2 (Biclique Coloring). *A coloring φ of a graph G is called a biclique coloring if for any two colors i and j, the subgraph induced by $N_i(C_j)$ and $N_j(C_i)$ is either empty or a complete bipartite graph.*

We will consider vertex cover in graphs for which a local biclique coloring φ is given. For any fixed $k \geq 3$, it is shown in [13] that there are n-vertex graphs with local chromatic number k and chromatic number $\Theta(\log \log n)$. This is shown using biclique colorings (cf. Def. 1.3 and Lemma 1.1 in [13]). Consequently, there are graphs that have chromatic number $\Theta(\log \log n)$ and admit a 3-local biclique coloring.

Contribution: In this paper we study the vertex cover problem in graphs with bounded local colorings. Assuming the UGC [23], the provided results give a tight characterization of the problem. The two main results are summarized as follows.

Theorem 1. *The vertex cover problem in graphs $G = (V, E)$ for which a $(\Delta+1)$-local biclique coloring φ of G is given as input admits a randomized polynomial-time algorithm with approximation ratio $2 - \Omega(1)\frac{\ln \ln \Delta}{\ln \Delta}$.*

Theorem 2. *Assuming the UGC, it is NP-hard to approximate the vertex cover problem in graphs for which a $(\Delta+1)$-local coloring is given as input, within any constant factor better than $2 - 2/(\Delta + 1)$.*

The result stated in Theorem 1 matches (up to the constant factor in the lower order term) a known inapproximability result [5]. In Section 3 we provide a matching upper bound for the inapproximability result of Theorem 2.

Besides generalizing the bounded degree case to a class of graphs with arbitrarily large chromatic number, we show that Theorem 1 finds a natural application in the precedence constrained single machine scheduling problem to minimize the weighted sum of completion times, known as $1|prec|\sum w_j C_j$ in standard scheduling notation. In a series of papers [1, 9, 10] it was established that this scheduling problem is a special case of minimum weighted vertex cover in graphs $G_{\mathbf{P}}$ of incomparable pairs defined in the dimension theory of partial orders. We prove the following in this paper.

Theorem 3. *For any graph of incomparable pairs $G_{\mathbf{P}}$, a $(\Delta + 1)$-local biclique coloring of $G_{\mathbf{P}}$ can be computed in polynomial time, where $\Delta - 1$ is the maximum number of predecessors (or successors) of each job.*

Together with Theorem 1, this improves the previously best $(2 - 2/\max\{\Delta, 2\})$-approximation algorithm described in [2]. Due to space limitations, omitted proofs will appear in the full version of the paper.

Review of the SDP Approach for Bounded Degree Graphs: In [22] the authors consider the problem of coloring graphs using semidefinite programming. Given a graph $G = (V, E)$ on n vertices, and a real number $k \geq 2$, a *vector k-coloring* [22] of G is an assignment of unit vectors $v_i \in \mathbb{R}^n$ to each vertex $i \in V$, such that for any two adjacent vertices i and j the dot product of their vectors satisfies the inequality $v_i \cdot v_j \leq -1/(k-1)$. They show that it is possible to check if a graph admits a vector k-coloring by using semidefinite programming. Moreover, they prove that vector k-colorable graphs have a "large" independent set when k is "small".

Theorem 4 ([22]). *For every integral $k \geq 2$, a vector k-colorable graph $G = (V, E)$ with maximum degree d has an independent set I of value $\Omega(\frac{w(V)}{d^{1-2/k}\sqrt{\ln d}})$.*

In [25], it is proved that the following program is a semidefinite relaxation of the vertex cover problem. Moreover, it can be solved within an additive error of $\varepsilon > 0$ in polynomial time in $\ln \frac{1}{\varepsilon}$ and n using the ellipsoid method.

$$\begin{array}{lll} \min \sum_{u=1}^{n} w_u \frac{1+v_0 \cdot v_u}{2} & & \\ \text{s.t. } (v_i - v_0)(v_j - v_0) = 0, & \{i, j\} \in E & \\ \| v_u \| = 1, & u \in V \cup \{0\}, v_u \in \mathbb{R}^{n+1} & \end{array} \quad (1)$$

Note that in an "integral" solution of (1) (corresponding to a vertex cover), vectors for vertices that are picked coincide with v_0, while the other vectors coincide with $-v_0$. It is shown in [25] that the integrality gap is $2 - \varepsilon$, for any $\varepsilon > 0$, i.e., for every $\varepsilon > 0$ there is a graph G_ε such that $vc(G_\varepsilon)/sd(G_\varepsilon)$ is at least $2 - \varepsilon$, where $vc(G_\varepsilon)$ and $sd(G_\varepsilon)$ denote the minimum vertex cover value and the optimum value of (1).

Halperin [19] uses (1) to provide an efficient randomized algorithm that approximates vertex cover in graphs with maximum degree d. The improvement is obtained as follows by using a threshold rounding approach. Solve relaxation (1) and let $S_1 = \{u \in V \mid v_0 \cdot v_u \geq x\}$ and $S_2 = \{u \in V \mid -x \leq v_0 \cdot v_u < x\}$, where x is a small positive number. As in Hochbaum's approach [20], it holds that the vertices S_1 together with a cover of the subgraph induced by S_2 are sufficient to cover the whole graph G. Moreover, for any two adjacent vertices i and j in the graph $G[S_2]$ induced by S_2, (1) implies that $v_i \cdot v_j \leq -1 + 2x$. This has the important consequence that $G[S_2]$ is a vector k-colorable graph, where $k = \frac{2-2x}{1-2x}$ is close to 2 for small x. We can then use[3] Theorem 4 to obtain a large valued independent set I of $G[S_2]$. The returned vertex cover is $S_1 \cup (S_2 \setminus I)$ and the result of [19] follows by choosing a suitable value for x.

The problem we consider in this paper is a classical problem in scheduling theory, known as $1|prec|\sum w_j C_j$ in standard scheduling notation (see e.g. Graham et al. [16]). It is defined as the problem of scheduling a set $N = \{1, \ldots, n\}$

[3] In Theorem 4 the integrality assumption on k is not technically necessary, and it can be easily generalized to fractional k. As remarked in [19], by using exactly the same analysis and rounding technique as in [22], it is possible to compute an independent set of value at least $\Omega(\frac{w(S_2)}{d^{x/(1-x)}\sqrt{x \ln d}})$ for $k = \frac{2-2x}{1-2x}$.

of n jobs on a single machine, which can process at most one job at a time. Each job j has a processing time p_j and a weight w_j, where p_j and w_j are nonnegative integers. Jobs also have precedence constraints between them that are specified in the form of a *partially ordered set* (*poset*) $\mathbf{P} = (N, P)$. The goal is to find a non-preemptive schedule which minimizes $\sum_{j=1}^{n} w_j C_j$, where C_j is the time at which job j completes in the given schedule.

The described problem was shown to be strongly NP-hard already in 1978 [27, 28]. For the general version of $1|prec|\sum w_j C_j$, several 2-approximation algorithms are known [32, 17, 9, 8, 29]. Until recently, no inapproximability results were known, and closing the approximability gap has been listed as one of ten outstanding open problems in scheduling theory (e.g., [33]). In [4], it is proved that the problem does not admit a PTAS, assuming that NP-complete problems cannot be solved in randomized subexponential time. Moreover, if a fixed cost present in all feasible schedules is ignored then the problem is as hard to approximate as vertex cover [4]. Recently, Bansal and Khot [6] showed that the gap for the general problem indeed closes assuming a variant of the UGC [23].

In a series of papers [1, 9, 10] it was proved that $1|prec|\sum w_j C_j$ is a special case of minimum weighted vertex cover in some special graphs $G_{\mathbf{P}}$ that depend on the input poset $\mathbf{P}$. More precisely, it is shown that any feasible solution to the vertex cover problem in graphs $G_{\mathbf{P}}$ can be turned in polynomial time into a feasible solution to $1|prec|\sum w_j C_j$ without deteriorating the objective value. This result was achieved by investigating different integer LP formulations and relaxations [31, 9, 10] of $1|prec|\sum w_j C_j$, using linear ordering variables δ_{ij} such that the variable δ_{ij} has value 1 if job i precedes job j in the corresponding schedule, and 0 otherwise.

Dushnik and Miller [12] introduced dimension as a parameter of partial orders in 1941. There is a natural way to associate with a poset $\mathbf{P}$ a hypergraph $\mathbf{H_P}$, called the *hypergraph of incomparable pairs*, so that the dimension of $\mathbf{P}$ is the chromatic number of $\mathbf{H_P}$ [15]. Furthermore, the fractional dimension of $\mathbf{P}$, a generalization due to Brightwell and Scheinerman [7] is equal to the fractional chromatic number of $\mathbf{H_P}$. It turns out [3] that graph $G_{\mathbf{P}}$ is the (ordinary) graph obtained by removing from $\mathbf{H_P}$ all edges of cardinality larger than two. This allows to apply the rich vertex cover theory to $1|prec|\sum w_j C_j$ together with the dimension theory of partial orders. One can, e.g., conclude that the scheduling problem with two-dimensional precedence constraints is solvable in polynomial time, as $G_{\mathbf{P}}$ is bipartite in this case [15, 10], and the vertex cover problem is well-known to be solvable in polynomial time on bipartite graphs. Further, these connections between the $1|prec|\sum w_j C_j$ and the vertex cover problem on $G_{\mathbf{P}}$, and between dimension and coloring, yield a framework for obtaining $(2-2/f)$-approximation algorithms for classes of precedence constraints with bounded (fractional) dimension f [2, 1]. It yields the best known approximation ratios for all previously considered special classes of precedence constraints, like semi-orders, convex bipartite orders, interval orders, interval dimension 2, bounded in-degree posets.

2 Vertex Cover Using Bounded Local Biclique Colorings

In this section we provide and analyze an approximation algorithm for the vertex cover problem in graphs $G = (V, E)$ for which a $(\Delta + 1)$-local biclique coloring φ of G is given.

The presented approximation algorithm follows the threshold rounding approach used for the bounded-degree vertex cover problem [19]: first solve the SDP (1) and, based on this solution and a parameter x that will be determined later, two vertex sets, S_1 and S_2, are computed as follows.

$$S_1 = \{u \in V \mid v_0 \cdot v_u \geq x\} \quad \text{and} \quad S_2 = \{u \in V \mid -x \leq v_0 \cdot v_u < x, \}$$

The cover is obtained by picking the vertices in S_1 together with a cover of the subgraph induced by S_2.

However, unlike in [19], Theorem 4 does not apply to graph $G[S_2]$. Indeed, the graph degree is not bounded and the theorem does not generalize to these graphs since the rounding procedure and analysis in [22] are strongly based on the assumption that the graph has "few", i.e. $O(n)$ edges.

We show that graph $G[S_2]$ has a "large" independent set by using a new rounding procedure to compute it that works as follows. The vertices in S_2 are first grouped into overlapping *clusters*. For every two colors i and j (of the given coloring φ) such that the number of edges connecting vertices with the two colors is non-zero, there are two clusters $N_i(C_j)$ and $N_j(C_i)$. Note that $N_j(C_i) \neq \emptyset$ iff $N_i(C_j) \neq \emptyset$ in $G[S_2]$. Because we assume that the coloring φ is a local $(\Delta + 1)$-coloring, each vertex $u \in V$ belongs to at most Δ clusters. Furthermore, for all i and j, clusters $N_i(C_j)$ and $N_j(C_i)$ are connected by complete bipartite sub-graphs (note that this property also holds when restricting the graph G to the vertex set S_2). It can be easily proved, that all vectors v_i corresponding to vertices from the same cluster almost point in the same direction. This essentially follows from having a biclique coloring and because, by the definition of the set S_2, vectors corresponding to adjacent vertices in S_2 are almost antipodal. For each cluster $N_i(C_j)$, we arbitrarily choose one *representative* vertex. Let R be the set of representatives of all clusters $N_i(C_j)$. Further, for a vertex u, let R_u be the set of representatives of clusters $N_i(C_j)$ for which $u \in N_i(C_j)$. By only using the vectors of the representatives, we compute a subset $I' \subseteq S_2$ as follows: vertex $u \in S_2$ belongs to I' if and only if every representative $a \in R_u$ of u satisfies $v_a \cdot r \geq c$, where c is a parameter and r is a random $(n+1)$-dimensional vector r from the $(n+1)$-dimensional standard normal distribution, i.e., the components of r are independent Gaussian random variables with mean 0 and variance 1. We show that the set I' has a large independent set. Formally, we have

$$I' = \left\{ u \in S_2 : \bigwedge_{a \in R_u} v_a \cdot r \geq c \right\}. \tag{2}$$

The complete procedure is summarized in Algorithm 1.

We first show that for all $u \in S_2$, the vectors corresponding to representatives of u's clusters point in almost the same direction as the vector v_u corresponding to vertex u.

1. Solve SDP (1)
2. Let $S_1 = \{u \in [n] \mid v_0 \cdot v_u \geq x\}$, $S_2 = \{u \in [n] \mid -x \leq v_0 \cdot v_u < x\}$.
3. Find an IS I in $G[S_2]$ as follows:
   (a) Compute the set R of representatives
   (b) Let $R_u = \big\{a \in R : \{u, a\} \subseteq N_i(C_j)$ for any two colors i and j of the given coloring $\varphi\big\}$
   (c) Choose a random vector r and define
   $$I' = \{u \in S_2 \mid \bigwedge_{a \in R_u} v_a \cdot r \geq c\}$$
   (d) Get I by removing one vertex of every edge in $G[I']$ from I'
4. Output the constructed vertex cover $S_1 \cup (S_2 \setminus I)$

Algorithm 1: Vertex Cover Approximation Algorithm

Lemma 1. *Let $u \in S_2$ be a vertex, $a \in R_u$ be the representative vertex of any cluster to which u belongs, and $x \in (0, 1)$. We have $v_u \cdot v_a \geq 1 - 8x + o(x)$.*

The expected size of the set I' is $\sum_{u \in S_2} \Pr[u \in I']$. Using (2), we have $\Pr[u \in I'] = \Pr\left[\bigwedge_{a \in R_u} v_a \cdot r \geq c\right]$. In the following we compute a lower bound on the above probability. The subsequent analysis uses some basic properties of the normal distribution. Let X be a standard normal random variable (X has mean 0 and variance 1). We use $\mathcal{N}(x)$ to denote the probability that X is at least x. Hence, $\mathcal{N}(x) := \Pr(X \geq x) = \int_x^\infty \phi(t)dt$, where $\phi(t) = e^{-t^2/2}/\sqrt{2\pi}$ is the density of X .

Lemma 2. *For any $u \in S_2$ and any constant $\gamma > 0$*

$$\Pr\left[\bigwedge_{a \in R_u} v_a \cdot r \geq c\right] \geq \mathcal{N}(\alpha c) - \Delta \cdot \mathcal{N}\left(\frac{c}{\beta\sqrt{x}}\right),$$

where $\alpha = \gamma + 1 + o(1)$ and $\beta = \frac{4+o(1)}{\gamma}$.

Proof. Assume, without loss of generality, that $v_u = (1, 0, \ldots, 0)$. By Lemma 1, for every $a \in R_u$ $v_u \cdot v_a \geq 1 - \delta$, where $\delta = 8x + o(x)$. Then the first component $v_{a,1}$ of $v_a = (v_{a,1}, v_{a,2}, \ldots, v_{a,n+1})$ must be at least $1-\delta$. Let $A = (1-\delta, 0, \ldots, 0)$ and $B_a = (0, v_{a,2}, \ldots, v_{a,n+1})$ for any $a \in R_u$.

Since v_a is a unit length vector, we have that $(1-\delta)^2 + \sum_{j=2}^{n+1} v_{a,j}^2 \leq 1$, which which gives an upper bound on the length of B_a:

$$\| B_a \|^2 = \sum_{j=2}^{n+1} v_{a,j}^2 \leq 2\delta - \delta^2 < 2\delta.$$

We then get

$$\Pr\left[\bigwedge_{a\in R_u} v_a\cdot r\geq c\right] \geq \Pr\left[\bigwedge_{a\in R_u}(A\cdot r\geq(\gamma+1)c\wedge B_a\cdot r\geq-\gamma c)\right]$$

$$= 1-\Pr\left[A\cdot r<(\gamma+1)c\vee\bigvee_{a\in R_u} B_a\cdot r<-\gamma c\right]$$

$$\geq 1-\Pr\left[A\cdot r<(\gamma+1)c\right]-\sum_{a\in R_u}\Pr\left[B_a\cdot r<-\gamma c\right]$$

$$= \Pr\left[\frac{A}{\|A\|}\cdot r\geq\frac{(\gamma+1)c}{\|A\|}\right]-\sum_{a\in R_u}\Pr\left[-\frac{B_a}{\|B_a\|}\cdot r>\frac{\gamma c}{\|B_a\|}\right]$$

$$\geq \mathcal{N}\left(\frac{(\gamma+1)c}{1-\delta}\right)-\Delta\cdot\mathcal{N}\left(\frac{\gamma c}{\sqrt{2\delta}}\right)$$

$$= \mathcal{N}(\alpha c)-\Delta\cdot\mathcal{N}\left(\frac{c}{\beta\sqrt{x}}\right).$$

The last inequality follows because the sum of k independent Gaussian random variables with variances $\sigma_1^2,\ldots,\sigma_k^2$ is a Gaussian random variable with variance $\sum_{i=1}^k\sigma_i^2$. Consequently, the the dot product of a unit vector with r is a standard normal random variable. □

The total weight of the vertices that are removed from I' is upper bounded by the weight of vertices of the edges in the graph $G[I']$ induced by I'. The next lemma bounds the probability that a vertex $u\in S_2$ is in I' and that u has a neighbor $u'\in S_2$ that is also in I'.

Lemma 3. *Consider a vertex $u\in S_2$. The probability that u as well as some neighbor u' of u are in I' is upper bound by*

$$\Pr\left[u\in I'\wedge\exists u'\in S_2:\{u,u'\}\in E\wedge u'\in I'\right]\leq\Delta\cdot\mathcal{N}\left(\frac{c}{\sqrt{x}}\right).$$

Based on Lemmas 2 and 3, we can now lower bound the expected size of the computed independent set I and we can thus obtain a bound on the expected approximation ratio of Algorithm 1.

Theorem 5. *Choosing $x=\frac{1-o(1)}{25}\cdot\frac{\ln\ln\Delta}{\ln\Delta}$, $c=(1+o(1))\cdot\sqrt{\frac{2x}{1-25x}\ln\Delta}$, and $\gamma=4$, Algorithm 1 has an expected approximation ratio of $2-\frac{2-o(1)}{25}\cdot\frac{\ln\ln\Delta}{\ln\Delta}$.*

3 Vertex Cover in Graphs with Bounded Local Chromatic Number

Consider the vertex cover problem in graphs G_Δ for which a local $(\Delta+1)$-coloring is given. Theorem 6 shows that the approximation ratio achievable from relaxation (1) is no better than $2-\frac{2}{\Delta+1}$ if only the bounded local colorability, but not the biclique condition holds.

Theorem 6. *For any fixed* $\Delta \geq 2$ *and* $\varepsilon > 0$*, there is a local* $(\Delta+1)$*-colorable graph* $G_{\Delta,\varepsilon}$ *for which*

$$\frac{vc(G_{\Delta,\varepsilon})}{sd(G_{\Delta,\varepsilon})} \geq 2 - \frac{2}{\Delta+1} - \varepsilon,$$

where $vc(G)$ *and* $sd(G)$ *denote the size of a minimum weighted vertex cover of* G *and the solution value for the corresponding SDP relaxation* (1)*, respectively.*

Under the Unique Game Conjecture, Khot and Regev [24] proved that vertex cover is NP-hard to approximate better than $2-\varepsilon$, for any $\varepsilon > 0$.

Theorem 7 ([24]). *Assuming the Unique Game Conjecture, for arbitrarily small constants* $\varepsilon, \delta > 0$*, there is a polynomial time reduction mapping a SAT formula* ϕ *to an* n*-vertex graph* G *such that if* ϕ *is satisfiable then* G *has an independent set of size* $(\frac{1}{2}-\varepsilon)n$ *and if* ϕ *is unsatisfiable, then* G *has no independent set of size* δn*.*

By using Theorem 7, the reduction and the analysis in the proof of Theorem 6 can be easily adapted to obtain the following conditional hardness.

Theorem 8. *Assuming the Unique Game Conjecture, it is NP-hard to approximate the vertex cover problem in graphs for which a local* $(\Delta+1)$*-coloring is given as input, within any constant factor better than* $2 - 2/(\Delta+1)$*.*

A matching upper bound can be obtained by showing that an independent set of value at least $\frac{\sum_{i=1}^{n} w_i}{\Delta+1}$ is computable in polynomial time. Indeed, the following result shows that Turan's theorem (as proved by Caro and Wei) for bounded degree graphs can be generalized to graphs with bounded local colorings.

Theorem 9. *For any weighted graph* $G = (V,E)$*, there exists an independent set of value at least* $\sum_{v \in V} \frac{w_v}{\Delta_v+1}$*, where* $\Delta_v = |\{c(u) : u \in N(v)\}|$ *is the number of different colors in the neighborhood of* $v \in V$ *of any proper coloring* c *of* G*.*

By using standard techniques, an upper bound that matches the lower bound of Theorem 2 can be obtained in deterministic polynomial time.

Corollary 1. *There exists a* $(2-\frac{2}{\Delta+1})$*-approximation algorithm for the vertex cover problem in graphs for which a local* $(\Delta+1)$*-coloring is given as input.*

4 The Scheduling Application

Problem $1|prec|\sum w_j C_j$ is a classical and fundamental problem in scheduling theory [33]. Its complexity certainly depends on the poset complexity In particular, the dimension of the input poset has been established to be an important parameter for the approximability of the problem [2, 1]. Another natural parameter of partial orders is given by the poset in- or out-degree [14], namely the job maximum number of predecessors or successors, respectively. One of the first NP-complete proofs [28] for $1|prec|\sum w_j C_j$ shows that the problem remains strongly NP-hard even if every job has at most two predecessors (or successors) in the poset. In [2], the authors present a $(2 - 2/\max\{\Delta, 2\})$-approximation algorithm, where $\Delta - 1$ is the minimum between the in- and the out-degree of the input poset. In this section, we show how to use Theorem 1 to improve this by showing how to compute a Δ-local biclique coloring of $G_{\mathbf{P}}$.

Consider a *partially ordered set* $\mathbf{P} = (N, P)$. When neither $(x, y) \in P$ nor $(y, x) \in P$, we say that x and y are incomparable, denoted by $x||y$. We call $\text{inc}(\mathbf{P}) = \{(x, y) \in N \times N : x||y \text{ in } P\}$ the set of *incomparable pairs* of $\mathbf{P}$. For any integer $k \geq 2$, a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subset \text{inc}(\mathbf{P})$ is called an *alternating cycle* when $x_i \leq y_{i+1}$ in P, for all $i = 1, 2, \ldots, k$, and where $y_{k+1} = y_1$. An alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ is *strict* if $x_i \leq y_j$ in P if and only if $j = i + 1$, for all $i, j = 1, 2, \ldots, k$.

For a poset $\mathbf{P}$, the *hypergraph of incomparable pairs of* $\mathbf{P}$ [15], denoted $\mathbf{H}_{\mathbf{P}} = (V, E)$, is the hypergraph that satisfies the following conditions: (1) V is the set $\text{inc}(\mathbf{P})$ of incomparable pairs of $\mathbf{P}$; and (2) E consists of those subsets of V that form strict alternating cycles. The *graph* $G_{\mathbf{P}}$ *of incomparable pairs of* $\mathbf{P}$ is the ordinary graph determined by all edges of size 2 in $\mathbf{H}_{\mathbf{P}}$ Hence, in $G_{\mathbf{P}}$, there is an edge between incomparable pairs (i, j) and (k, ℓ) if an only if $(i, \ell), (k, j) \in P$.

In a series of papers [1, 10, 9], it was proved that $1|prec|\sum w_j C_j$ is equivalent to a weighted vertex cover problem on the graph of incomparable pairs $G_{\mathbf{P}}$ of the poset $\mathbf{P}$ characterizing the precedence constraints of the scheduling problem. More precisely, given a scheduling instance S with precedence constraints $\mathbf{P}$, we need to consider the following *weighted* version $G^S_{\mathbf{P}}$ of $G_{\mathbf{P}}$. For all incomparable pairs $(i, j) \in \text{inc}(\mathbf{P})$, the weight of vertex (i, j) in $G_{\mathbf{P}}$ is $p_i \cdot w_j$, where p_i is the processing time of job i and w_j is the weight of process j.

Let $\mathbf{P} = (N, P)$ be a poset. For $j \in N$, define the *degree of* j $\deg(j)$ as the number of elements comparable (but not equal) to j in $\mathbf{P}$. Given $j \in N$, let $D(j)$ denote the set of all elements which are less than j, and $U(j)$ those which are greater than j in P. Let $\deg_D(j) := |D(j)|$ be the *in-degree* of j and the *maximum in-degree* $\Delta_D(\mathbf{P}) := \max\{\deg_D(j) : j \in N\}$. The *out-degree* of j $\deg_U(j)$ and the *maximum out-degree* $\Delta_U(\mathbf{P})$ are defined analogously (see also [14]). The maximum vertex degree in the graph of incomparable pairs $G_{\mathbf{P}}$ is bounded by $(\Delta_D(\mathbf{P}) + 1) \cdot (\Delta_U(\mathbf{P}) + 1)$. Hence, if both $\Delta_D(\mathbf{P})$ and $\Delta_U(\mathbf{P})$ are bounded, $G_{\mathbf{P}}$ has bounded degree and therefore, the bounded degree vertex cover approximation of [19] can be used to approximate the scheduling problem $1|prec|\sum w_j C_j$ with precedence constraints $\mathbf{P}$. If only either the in-degree or the out-degree of $\mathbf{P}$ is bounded, $G_{\mathbf{P}}$ does not have bounded degree. However, we will now show that in this case $G_{\mathbf{P}}$ has a good local biclique coloring.

Theorem 10. *Let* $\mathbf{P} = (N, P)$ *be a poset and let* $\Delta = 1 + \min\{\Delta_D(\mathbf{P}), \Delta_U(\mathbf{P})\}$. *Then, we can efficiently compute a* $(\Delta+1)$*-local biclique coloring of* $G_{\mathbf{P}} = (V, E)$.

Proof. We assume that $\Delta - 1 = \Delta_D(\mathbf{P})$ is the largest in-degree. The case $\Delta - 1 = \Delta_U(\mathbf{P})$ can be proven analogously. We first show how to compute a $(\Delta+1)$-local coloring of $G_{\mathbf{P}}$ (cf. Def. 1). Partition the incomparable pairs into $|N|$ color classes: $C_i = \{(i, j) \in \text{inc}(\mathbf{P})\}$ for $i \in [|N|]$. It is easy to check that every C_i forms an independent set. Moreover any incomparable pair $(i, j) \in \text{inc}(\mathbf{P})$ is adjacent to $(k, \ell) \in \text{inc}(\mathbf{P})$ if (necessary condition) $(k, j) \in P$. Since the in-degree of j is bounded by $\Delta - 1$, it follows that the number of distinct pairs (k, j) such that $(k, j) \in P$ is bounded by Δ (it is Δ and not $\Delta - 1$ because we also have to consider the pair $(j, j) \in P$). Therefore any incomparable pair (i, j) has neighbors in at most Δ clusters and thus the coloring is $(\Delta + 1)$-local.

In order to show that the coloring is a biclique coloring (cf. Def. 2), assume that $\{(i, a), (j, c)\} \in E$ and $\{(i, b), (j, d)\} \in E$. The claim follows by proving that $\{(i, a), (j, d)\} \in E$ and $\{(i, b), (j, c)\} \in E$. By the assumption we have: $(i, c) \in P$, $(j, a) \in P$, $(i, d) \in P$ and $(j, b) \in P$. Since $(j, a) \in P \wedge (i, d) \in P$ we have $\{(i, a), (j, d)\} \in E$, and $(i, c) \in P \wedge (j, b) \in P$ implies $\{(i, b), (j, c)\} \in E$. □

Acknowledgments. The second author is indebted to Nikos Mutsanas and George Karakostas for many interesting discussions. We would like to thank Jiři Sgall for suggesting the term "biclique coloring." The research is supported by the Swiss National Science Foundation project 200020-122110/1 and by Hasler Foundation Grant 11099.

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