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Network Algorithms, Summer Term 2012
Problem Set 8

hand in by Wednesday, July 25 , 2012, 12:00

For this exercise, you may use the following inequalities, based on the *Chernoff bound.*[1]

Theorem 1 (Chernoff Bounds) *Let $X = \sum_{i=1}^{N} X_i$ be the sum of N independent $0-1$ random variables X_i.*
Bound 1:

$$\Pr\left[|X - \mathbb{E}[X]| \leq \lambda\left(\log n + \sqrt{\mathbb{E}[X]\log n}\right)\right] > 1 - \frac{1}{n^c},$$

where the constant c can be made arbitrarily large by choosing a sufficiently large constant λ.
Bound 2:

$$\Pr\left[X \leq (1-\delta)\mathbb{E}[X]\right] \leq e^{-\mathbb{E}[X]\delta^2/2}$$

for all $0 < \delta \leq 1$.

Exercise 1: Diameter of the Augmented Grid

Recall the network from the lecture where nodes were arranged in a grid and each node had an additional directed link to an uniformly and independently at random drawn node in the network (i.e., $\alpha = 0$). In the lecture, a proof of the fact that such a network has diameter $O(\log n)$ w.h.p. was sketched. We will now fill in the details.

a) Show that $O(n/\log n)$ many nodes are enough to guarantee with high probability that at least one of their random links connects to a given set of $\Omega(\log^2 n)$ nodes. Prove this (i) by direct calculation and (ii) using Chernoffs bound.

   Hint: use that $1 - p \leq e^{-p}$ for any p.

b) Suppose for some node set S we have that $|S| \in \Omega((\log n)^2) \cap o(n)$ and denote by H the set of nodes hit by their random links. Prove that H and together with its grid neighbors contains w.h.p., $(5o(1))|S|$ nodes!

   Hint: Observe that *independently* of all previous random choices, each new link has at least a certain probability p of connecting to a node whose complete neighborhood has not been reached yet. Then use Chernoffs bound on the sum of $|S|$ many variables.

c) Infer from b) that starting from $\Omega(\log^2 n)$ nodes, with each hop the number of reached nodes w.h.p. more than doubles, as long as we have still $O(n/\log n)$ nodes (regardless of the constants in the O-notation).

   Hint: Play with the constant c in Bound 1 of Theorem 1 and use the union bound.

d) Conclude that the diameter of the network is w.h.p. in $O(\log n)$.

[1] Chernoff-type and similar probability bounds are very powerful tools that allowed to design a plethora of randomized algorithms that *almost* guarantee success. Frequently this "almost" makes a huge difference in e.g., running time and/or approximation quality.