Exercise 1: \( \mathcal{NP} \) and Star Operation (5 points)

Show that \( \mathcal{NP} \) is closed under the star operation.

Remark 1: Let \( A \) be a language. The operation \( \text{star}(\cdot^*) \) is defined as follows:
\[
A^* = \{ x_1 x_2 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in A \text{ where } 0 \leq i \leq k \}.
\]

Remark 2: A collection of objects is closed under some operation if applying that operation (a finite number of times) to members of the collection returns an object still in the collection.

Exercise 2: The class \( \mathcal{NPC} \) (8 points)

Let \( L_1, L_2 \) be languages (problems) over alphabets \( \Sigma_1, \Sigma_2 \). Then \( L_1 \leq_p L_2 \) (\( L_1 \) is polynomially reducible to \( L_2 \)), iff a function \( f : \Sigma_1^* \rightarrow \Sigma_2^* \) exists, that can be calculated in polynomial time and
\[
\forall s \in \Sigma_1 : s \in L_1 \iff f(s) \in L_2.
\]
Language \( L \) is called \( \mathcal{NP} \)-hard, if all languages \( L' \in \mathcal{NP} \) are polynomially reducible to \( L \), i.e.
\[
L \text{ \( \mathcal{NP} \)-hard} \iff \forall L' \in \mathcal{NP} : L' \leq_p L.
\]
The reduction relation \( \leq_p \) is transitive (\( L_1 \leq_p L_2 \) and \( L_2 \leq_p L_3 \) \( \Rightarrow \) \( L_1 \leq_p L_3 \)). Therefore, in order to show that \( L \) is \( \mathcal{NP} \)-hard, it suffices to reduce a known \( \mathcal{NP} \)-hard problem \( \bar{L} \) to \( L \), i.e. \( \bar{L} \leq_p L \).

Finally a language is called \( \mathcal{NP} \)-complete (\( \Leftrightarrow \): \( L \in \mathcal{NPC} \)), if
1. \( L \in \mathcal{NP} \) and
2. \( L \) is \( \mathcal{NP} \)-hard.

Show \text{HittingSet} := \{ \langle U, S, k \rangle \mid \text{universe } U \text{ has subset } H, |H| \leq k \text{ that hits all sets in } S \subseteq 2^U \} \in \mathcal{NPC}.$1$

Use that \text{VertexCover} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k \} \in \mathcal{NPC}.

Remark: A hitting set \( H \subseteq U \) for a given universe \( U \) (which is a finite set) and a set \( S = \{ S_1, S_2, \ldots, S_m \} \) of subsets \( S_i \subseteq U \), fulfills the property \( H \cap S_i \neq \emptyset \) for \( 1 \leq i \leq m \) (\( H \) ‘hits’ at least one element of every \( S_i \)).

A vertex cover is a subset \( V' \subseteq V \) of nodes of \( G = (V, E) \) such that every edge of \( G \) is adjacent to a node in the subset.

Hint: For the poly. transformation (\( \leq_p \)) you have to describe an algorithm (with poly. run-time!) that transforms an instance \( \langle G, k \rangle \) of \text{VertexCover} into an instance \( \langle U, S, k \rangle \) of \text{HittingSet}, s.t. a vertex cover of size \( \leq k \) in \( G \) becomes a hitting set of \( U \) of size \( \leq k \) for \( S \) and vice versa(!).

---

$1$The power set \( 2^U \) of some ground set \( U \) is the set of all subsets of \( U \). So \( S \subseteq 2^U \) is a collection of subsets of \( U \).
Exercise 3: Complexity Classes: Big Picture (2+3+2 points)

(a) Why is $P \subseteq NP$?

(b) Show that $P \cap NPC = \emptyset$ if $P \neq NP$.
   
   Hint: Assume that there exists a $L \in P \cap NPC$ and derive a contradiction to $P \neq NP$.

(c) Give a Venn Diagram showing the sets $P, NP, NPC$ for both cases $P \neq NP$ and $P = NP$.
   
   Remark: Use the results of (a) and (b) even if you did not succeed in proving those.