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Theoretical Computer Science - Bridging Course
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Exercise Sheet 10

Hand in (electronically or hard copy) by 12:15 pm, July 24, 2017

Exercise 1: Completeness and Correctness of Calculi (2+1+1 points)

A calculus $\mathbf{C}$ is called *correct* if for every knowledge base KB and formula φ the following holds

$$KB \vdash_{\mathbf{C}} \varphi \quad \Longrightarrow \quad KB \models \varphi.$$

Calculus $\mathbf{C}$ is called *complete* if

$$KB \models \varphi \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \varphi.$$

Remark: For the definition of '$\models$' consult Exercise Sheet 9 or the lecture.

Consider the following calculi

$$\mathbf{C_1}: \quad \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi} \qquad \mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi} \qquad \mathbf{C_3}: \quad \frac{\varphi, \psi \to \varphi}{\psi}$$

(a) Show that the $\mathbf{C_1}$ and $\mathbf{C_2}$ are both correct. *Hint: Use truth tables.* Give a short explanation why $\mathbf{C_1}, \mathbf{C_2}$ are correct.

(b) Show that $\mathbf{C_3}$ is not correct. *Hint: Use a truth table*

(c) Show that $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C_3}$ are not complete by giving a knowledge base KB and a formula φ such that $KB \models \varphi$ but not $KB \vdash_{\mathbf{C_i}} \varphi$.

Exercise 2: Resolution (1+2+3 points)

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

Thus, in order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus $\mathbf{C}$ is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Therefore, if we have a refutation-complete calculus $\mathbf{C}$, it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ in order to prove $KB \models \varphi$.

The *Resolution Calculus*[1] **R** is correct and refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \dots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \dots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$
Remark: KB represents the formula $C_1 \wedge \dots \wedge C_n$ with $C_i = L_{i,1} \vee \dots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L, \neg L\} \vdash_{\mathbf{R}} \bot$).

(a) We want to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$.

(c) Consider the sentence "Heads, I win". "Tails, you lose". Design a propositional KB that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that **I always win**.

Exercise 3: Predicate Logic: Construct Formulae (1+1+1+1 points)

Let $\mathcal{S} = \langle \{x, y, z\}, \emptyset, \emptyset, \{R\} \rangle$ be a signature. Translate the following sentences of first order formula over $\mathcal{S}$ into idiomatic English. Use $R(x, y)$ as statement "x is a part of y".

(a) $\exists x \forall y R(x, y)$.

(b) $\exists y \forall x R(x, y)$.

(c) $\forall x \forall y \exists z (R(x, z) \wedge R(y, z))$

(d) $\forall x \exists y (R(y, x) \wedge \neg \exists z (R(z, y) \wedge \neg R(y, z)))$

Exercise 4: Predicate Logic: Entailment (2+2+2 points)

Let φ, ψ be first order formulae over signature $\mathcal{S}$. Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of φ is also a model for ψ. We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base* KB is a set of formulae. A model of KB is model for all formulae in KB. We write $KB \models \varphi$ if all models of KB are models of φ. Show or disprove the following entailments.

(a) $(\exists x\, R(x)) \wedge (\exists x\, P(x)) \wedge (\exists x\, T(x)) \models \exists x\, (R(x) \wedge P(x) \wedge T(x))$.

(b) $(\forall x \forall y\, f(x, y) \doteq f(y, x)) \wedge (\forall x\, f(x, \mathbf{c}) \doteq x) \models \forall x\, f(\mathbf{c}, x) \doteq x$.

(c) $(\forall x\, R(x, x)) \wedge (\forall x \forall y\, R(x, y) \wedge R(y, x) \rightarrow x \doteq y) \wedge (\forall x \forall y \forall z\, R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
$\models \forall x \forall y\, R(x, y) \vee R(y, x)$.

Hint: Consider order relations. E.g., $a \leq b$ (a less-equal b) and $a|b$ (a divides b).

[1] Complete calculi are unpractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.