Albert-Ludwigs-Universität, Inst. für Informatik
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Theoretical Computer Science - Bridging Course
Winter Term 2016
Exercise Sheet 12

Hand in (electronically or hard copy) before your weekly meeting but not later than 23:59, Wednesday, February 1, 2016

Exercise 1: Predicate Logic: Interpretations (2+2+2+2 points)

In *predicate logic* or *first order* a formula φ is given with respect to a *signature* $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{R} \rangle$ which introduces the basic components that φ consists of. The components are: The variable symbols $\mathcal{V}$, constant symbols $\mathcal{C}$, function symbols $\mathcal{F}$ and the set of relation symbols $\mathcal{R}$. The elements of the sets $\mathcal{V}, \mathcal{C}, \mathcal{F}$ are used to formulate *terms*, while the relations in $\mathcal{R}$ compare terms with each other.[1]

These components must be combined in a *well-formed* manner with logical connectives ($\wedge, \vee, \neg$, etc.), quantifiers ($\forall, \exists$), and the '$\doteq$' symbol which is a relation that represents equality of terms. Consult the lecture for the detailed inductive definition of first order formulae.

As it was the case in propositional logic, we require *interpretations* in order to evaluate first order formulae to true or false. An interpretation $I = \langle \mathcal{D}, \cdot^I \rangle$ has a *domain* $\mathcal{D}$ that represents the set of all values that variables can assume.

Furthermore an interpretation has a mapping $\cdot^I$ which assigns constant symbols $\mathbf{c} \in \mathcal{C}$ a fixed value $\mathbf{c}^I \in \mathcal{D}$ from the domain. A function symbol $f \in \mathcal{F}$ is assigned an explicit function $f^I : \mathcal{D}^k \to \mathcal{D}$. A relation symbol $R \in \mathcal{R}$ is assigned an explicit relation $R^I \subseteq \mathcal{D}^k$. The parameter k is called *arity*.

If a formula φ has *free variables* (variables that are not bound by a quantifier: $\forall, \exists$) then I requires an additional variable *assignment function* $\alpha : \mathcal{V} \to \mathcal{D}$ assigning each free variable a value from the domain. An interpretation I is called a model of a first order formula φ, if an assignment function α *exists* (!) such that $\varphi^{I,\alpha}$ evaluates to true (see lecture for details on how to evaluate φ with I, α).

Evaluate the given formulae with the given interpretations. Make clear why or why not an interpretation is a model for the formula.

(a) $\varphi_1 := \forall x \exists y\, f(y) \doteq x$, $I_1 := \langle \mathbb{Z}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{Q}, \cdot^{I_2} \rangle$ where $f^{I_1}(a) := f^{I_2}(a) := 2 \cdot a$.

(b) $\varphi_2 := \forall x \exists y\, f(y, y) \doteq x$, $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{C}, \cdot^{I_2} \rangle$ where $f^{I_1}(a, b) := f^{I_2}(a, b) := a \cdot b$.

(c) $\varphi_3 := (\forall x\, f(x, z) \doteq x) \wedge (\forall x \exists y\, f(x, y) \doteq z)$, $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{R}, \cdot^{I_2} \rangle$ where $f^{I_1}(a, b) := a + b, f^{I_2}(a, b) := a \cdot b$. *Hint: First determine which z satisfies the first condition.*

(d) $\varphi_4 := \forall x \forall y \forall z\, f(x, f(y, z)) \doteq f(f(x, y), z)$. Give a model and an interpretation that is no model.

[1] In the following the signature is given implicitly via the symbols that are given in a formula. By convention bold symbols $\mathbf{c}$ represent constants, lowercase letters f functions and capital letters R relations.

Exercise 2: Predicate Logic: Construct Formulae (1+1+1+1 points)

Let $\mathcal{S} = \langle \{x\}, \emptyset, \emptyset, \{P, Q, R, S\} \rangle$ be a signature. Express each of the following statements as first order formula over $\mathcal{S}$. Use $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ as statements "x is a duck", "x is one of my poultry", "x is an officer", and "x is willing to waltz" respectively.

(a) No ducks are willing to waltz.

(b) No officers ever decline to waltz.

(c) All my poultry are ducks.

(d) My poultry are not officers.

Exercise 3: Predicate Logic: Entailment (2+2+2+2 points)

Let φ, ψ be first order formulae over signature $\mathcal{S}$. Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of φ is also a model for ψ. We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base* KB is a set of formulae. A model of KB is model for all formulae in KB. We write $KB \models \varphi$ if all models of KB are models of φ. Show or disprove the following entailments.

(a) Let KB be the formulae derived in 2 a), b), c) and φ the one derived in d). Show $KB \models \varphi$.

(b) $(\exists x\, R(x)) \wedge (\exists x\, P(x)) \models \exists x\, R(x) \wedge P(x)$.

(c) $(\forall x \forall y\, f(x, y) \doteq f(y, x)) \wedge (\forall x\, f(x, \mathbf{c}) \doteq x) \models \forall x\, f(\mathbf{c}, x) \doteq x$.

(d) $(\forall x\, R(x, x)) \;\wedge\; (\forall x \forall y\, R(x, y) \wedge R(y, x) \rightarrow x \doteq y) \;\wedge\; (\forall x \forall y \forall z\, R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
$\models \forall x \forall y\, R(x, y) \vee R(y, x)$.

Hint: Consider order relations. E.g., $a \leq b$ (a less-equal b) and $a|b$ (a divides b).