Albert-Ludwigs-Universität, Inst. für Informatik
Prof. Dr. Fabian Kuhn
M. Ahmadi, P. Bamberger, H. Ghodselahi, Y. Maus, P. Schneider May 5, 2017

Theoretical Computer Science - Bridging Course
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Sample Solution for Exercise Sheet 1

Exercise 1: Proof by Induction (5 points)

Prove the famous Gaussian summation formula by induction on n:

For all natural numbers $n \geq 1$ it holds: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Solution:

For $n = 1$ we have to show that $\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$, which is obviously true because both sides are equal 1.

Now assume the statement is true for n. It follows that

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

which shows that the statement also holds for $n+1$. The second equality in the equation above comes from the assumption for n.

Exercise 2: Partition of a Set (5 points)

A partition of a set A is a collection of sets $B_i, i \in \{1, \dots, n\}$ such that

$$B_1 \cup \ldots \cup B_n = A \text{ and } B_i \cap B_j = \emptyset \text{ for } i \neq j.$$

Show that $B_i := \{3k + i \,|\, k \in \mathbb{Z}\}$, $i \in \{1, 2, 3\}$ is a partition of $\mathbb{Z}$.

Hint: $\mathbb{Z}$ is the set of integers. In order to proof that two sets are equal consider an arbitrary element from one set and show that it is contained in the other set and vice versa.

Solution:

First we show $B_1 \cup B_2 \cup B_3 = \mathbb{Z}$. Let z be an arbitrary element in $B_1 \cup B_2 \cup B_3$. This means $z \in B_i$ for at least one of the B_i. Since B_i is composed only of integers we conclude $z \in \mathbb{Z}$.
Now let $z \in \mathbb{Z}$ be an arbitrary integer and $i := z \bmod 3 \leq 2$ the residue of the integer division of z by three. Then we have $z = 3k + i$ for some $k \in \mathbb{Z}$ and thus either $z \in B_i$ for $i = 1, 2$ or $z \in B_3$ for $i = 0$. Therefore $z \in B_1 \cup B_2 \cup B_3$.
It remains to be shown that the intersection $B_i \cap B_j$ for $i \neq j$ is empty. For a contradiction assume there exists a $z \in B_i \cap B_j$. Since $z \in B_i$ and $z \in B_j$ we have (by definition of B_1, B_2, B_3) that $i = z \bmod 3 = j$ which is a contradiction.

Exercise 3: Counting Edges in Acyclic Graph (5 points)

A tree is an acyclic, connected, simple graph. Show that a tree with $n \geq 1$ nodes has $n-1$ edges. A forest is a graph consisting of several unconnected trees. Show that a forest consisting of k components has $n-k$ edges.

Hint: A simple graph is an unweighted, undirected graph containing no self-loops or multiple edges.

Solution:

First we show that an acyclic, connected graph with n nodes has exactly $n-1$ edges using an induction argument on n. A graph with just one node has $n-1=0$ edges. Assume that the statement holds for graphs with an arbitrary but fixed number of nodes n and consider a graph G with $n+1$ nodes. We remove one edge e, which makes G disintegrate into two components G_1 and G_2 which are not connected to each other (if there were a connection between G_1 and G_2, then reattaching e to G would close a cycle).
The components G_1 and G_2 themselves are still acyclic and (internally) connected and have $1 \leq k, m \leq n$ nodes with $k+m=n+1$. Using the induction hypothesis ($k, m \leq n$) we have that G_1 has $k-1$ edges and G_2 has $m-1$ edges. Since we removed exactly one edge to obtain G_1 and G_2, G has $(k-1)+(m-1)+1=n$ edges.
Next we show that a forest G consisting of k trees $G_1, \ldots, G_k$ has $n-k$ edges. Let n_i, $i \in \{1, \ldots, k\}$ the number of nodes of the i-th tree. Of course $\sum_{i=1}^{k} n_i = n$. We already know that G_i has $n_i - 1$ edges. Thus G has exactly $\sum_{i=1}^{k} n_i - 1 = n-k$ edges.

Exercise 4: Nodes with Identical Degrees (5 points)

Show that every simple graph with two or more nodes contains two nodes with the same degree.

Solution:

We prove this claim by contradiction. Consider a graph with $n \geq 2$ nodes $u_1, u_2, \ldots, u_n$. Assume that each node has a different degree. The minimum degree a node can have is 0, in which case the node has no neighbours; and the maximum degree a node can have is $n-1$, in which case the node connects to every other node in the graph. Without loss of generality, we assume that node u_i has degree $i-1$, where $1 \leq i \leq n$ (otherwise we rename nodes). Since node u_n has degree $n-1$ it must be connected to all others including u_1. However, the degree of u_1 is 0, which is a contradiction.

Alternative approach. You can also prove this by induction on n. However, in this process, you still may have to use the trick we employed in the above proof: counting degrees carefully.