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Theoretical Computer Science - Bridging Course
Summer Term 2017
Exercise Sheet 9 - Sample Solution

Exercise 1: Propositional Logic: Basic Terms (2+2+2 points)

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \leftrightarrow q) \leftrightarrow (r \leftrightarrow \neg p)$

(b) $\varphi_2 = (p \to q) \not\to ((\neg p \to q) \to r)$

(c) $\varphi_3 = (p \wedge q) \to (p \vee r)$

Remark: $a \to b :\equiv \neg a \vee b$, $a \leftrightarrow b :\equiv (a \to b) \wedge (b \to a)$, $a \not\to b :\equiv \neg(a \to b)$.

Sample Solution

With truth tables it is easy to check whether the at most $2^3 = 8$ different interpretations fulfill the above formulae. Note the pattern of 0 ($=F$) and 1 ($=T$) we use to obtain all possible interpretations.

(a) See Table 1. The result shows that φ_1 is satisfiable.

(b) See Table 2. The result shows that φ_2 is satisfiable.

(c) See Table 3. The result shows that φ_3 is a tautology.

model	p	q	r	$p \leftrightarrow q$	$r \leftrightarrow \neg p$	φ_1
✗	0	0	0	1	0	0
✓	0	0	1	1	1	1
✓	0	1	0	0	0	1
✗	0	1	1	0	1	0
✗	1	0	0	0	1	0
✓	1	0	1	0	0	1
✓	1	1	0	1	1	1
✗	1	1	1	1	0	0

Table 1: Truthtables for Exercises 1 (a).

model	p	q	r	$p \to q$	$\neg p \to q$	$(\neg p \to q) \to r$	φ_2
✗	0	0	0	1	0	1	0
✗	0	0	1	1	0	1	0
✓	0	1	0	1	1	0	1
✗	0	1	1	1	1	1	0
✗	1	0	0	0	1	0	0
✗	1	0	1	0	1	1	0
✓	1	1	0	1	1	0	1
✗	1	1	1	1	1	1	0

Table 2: Truthtables for Exercises 1 (b).

model	p	q	r	$p \wedge q$	$p \vee r$	φ_3
✓	0	0	0	0	0	1
✓	0	0	1	0	1	1
✓	0	1	0	0	0	1
✓	0	1	1	0	1	1
✓	1	0	0	0	1	1
✓	1	0	1	0	1	1
✓	1	1	0	1	1	1
✓	1	1	1	1	1	1

Table 3: Truthtables for Exercises 1 (c).

Exercise 2: CNF and DNF (2+2 points)

(a) Convert $\varphi_1 := (p \to q) \to (\neg r \land q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \to \neg q) \land \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Sample Solution

(a)

$$\begin{aligned}
& (p \to q) \to (\neg r \land q) & \\
\equiv\ & \neg(\neg p \lor q) \lor (\neg r \land q) & \text{Definition of '}\to\text{'} \\
\equiv\ & (p \land \neg q) \lor (\neg r \land q) & \text{De Morgan} \\
\equiv\ & \big((p \land \neg q) \lor \neg r\big) \land \big((p \land \neg q) \lor q\big) & \text{Distribution} \\
\equiv\ & \big((p \lor \neg r) \land (\neg q \lor \neg r)\big) \land \big((p \lor q) \land (\neg q \lor q)\big) & \text{Distribution} \\
\equiv\ & \big((p \lor \neg r) \land (\neg q \lor \neg r)\big) \land \big((p \lor q) \land 1\big) & \text{Complementation} \\
\equiv\ & \big((p \lor \neg r) \land (\neg q \lor \neg r)\big) \land (p \lor q) & \text{Identity} \\
\equiv\ & (p \lor \neg r) \land (\neg q \lor \neg r) \land (p \lor q) & \text{Associativity}
\end{aligned}$$

(b)

$$\begin{aligned}
& \neg((\neg p \to \neg q) \land \neg r) & \\
\equiv\ & \neg((p \lor \neg q) \land \neg r) & \text{Definition of '}\to\text{'} \\
\equiv\ & \neg(p \lor \neg q) \lor r & \text{De Morgan} \\
\equiv\ & (\neg p \land q) \lor r & \text{De Morgan}
\end{aligned}$$

Exercise 3: Logical Entailment (2+2 points)

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \vee q, \neg r \vee p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b) $\varphi_2 := (q \leftrightarrow r) \rightarrow p$

Sample Solution

(a) KB does not entail φ_1. Consider the interpretation $I : p \mapsto 1, q \mapsto 0, r \mapsto 0$. Interpretation I is a model for KB but not for φ_1.

(b) Table 4 shows that every model of KB is also a model of φ_2, hence $KB \models \varphi_2$.

model of KB	p	q	r	$p \vee q$	$\neg r \vee p$	$q \leftrightarrow r$	φ_2	model of φ_2
✗	0	0	0	0	0	1	0	✗
✗	0	0	1	0	0	0	1	✓
✗	0	1	0	0	0	0	1	✓
✗	0	1	1	0	0	1	0	✗
✗	1	0	0	0	1	1	1	✓
✗	1	0	1	0	0	0	1	✓
✓	1	1	0	1	1	0	1	✓
✗	1	1	1	1	0	1	1	✓

Table 4: Truthtable for Exercise 3 (b).

Exercise 4: Inference Rules and Calculi (3+3 points)

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \land \psi}{\varphi, \psi}, \frac{(\varphi \land \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \land q, p \to r, (q \land r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Sample Solution

(a) We use $\mathbf{C_1}$ to derive new formulae until we obtain the desired one.

$$\begin{aligned}
\neg q \to r &\overset{\text{2nd rule}}{\vdash_{\mathbf{C_1}}} \neg r \to q \\
p \leftrightarrow \neg r &\overset{\text{3rd rule}}{\vdash_{\mathbf{C_1}}} p \to \neg r, \neg r \to p \\
p \to \neg r, \neg r \to q &\overset{\text{1st rule}}{\vdash_{\mathbf{C_1}}} p \to q
\end{aligned}$$

(b) We use $\mathbf{C_2}$ to derive new formulae until we obtain the desired one.

$$\begin{aligned}
p \land q &\overset{\text{2nd rule}}{\vdash_{\mathbf{C_2}}} p, q \\
p, p \to r &\overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} r \\
(q \land r) \to s &\overset{\text{3rd rule}}{\vdash_{\mathbf{C_2}}} q \to (r \to s) \\
q, q \to (r \to s) &\overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} r \to s \\
r, r \to s &\overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} s
\end{aligned}$$