Exercise 1: Decidable Languages

(a) Show $L_1 := \{ \langle A \rangle \mid A \text{ is a CFG that generates } \varepsilon \}$ is decidable. Use that $A_{\text{CFG}} := \{ \langle A, w \rangle \mid A \text{ is a CFG that generates } w \}$ is decidable (cf. lecture).

(b) Show $L_2 := \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \Sigma^* \}$ is decidable. Use that $EQ_{\text{DFA}} := \{ \langle A, B \rangle \mid A, B \text{ are DFAs and } L(A) = L(B) \}$ is decidable (cf. lecture).

(c) Show that $\{ \langle R, S \rangle \mid R, S \text{ are reg. expr. and } L(R) \subseteq L(S) \}$ is decidable. Use that we can derive a DFA $A$ from a regular expression $R$ such that $L(A) = L(R)$ in finite time. Moreover for DFAs $A,B$ we can determine a DFA $C$ with $L(C) = L(A) \cap L(B)$ in finite time.
Exercise 2: Decidability

Consider the following language

\[ L := \begin{cases} 
\{0\}, & \text{if there is life on Mars} \\
\{1\}, & \text{else.} 
\end{cases} \]

Is \( L \) decidable?
Exercise 3: The Halting Problem

(a) Show that the Halting problem

\[ H := \{ \langle M, s \rangle \mid \langle M \rangle \text{ encodes a TM that halts on input } s \} \]

is semi-decidable.\(^1\)

(b) Show that the Halting problem \( H \) is undecidable. Assume that we already know that the Special Halting problem

\[ H_s := \{ \langle M \rangle \mid \langle M \rangle \text{ encodes a TM that halts on input } \langle M \rangle \} \]

is undecidable.

\(^1\)This definition of the halting problem deviates from the one on the lecture slides, but is also very common.
Exercise 4: Relation between Language Classes

(a) Give a Venn diagram showing the relation between the set of regular, context-free, decidable and semi-decidable languages.

(b) Give an explanation why some of these sets are contained in others.

(c) Show that the subset relations are proper, e.g. by giving a language which is contained in the respective superset but not in the subset.
Exercise 5: Conway’s Game of Life

Given a square grid $G = \mathbb{Z} \times \mathbb{Z}$, a cell $(x, y) \in G$ is either alive or dead. Each time step a cell changes its state according to the following rules:

1. A live cell with less than two live neighbors dies.
2. A live cell with two or three live neighbors lives on.
3. A live cell with four or more live neighbors dies.
4. A dead cell with exactly three neighbors becomes alive.

Is Conway’s game, which asks for a given start pattern $p_{\text{init}} : G \rightarrow \{\text{dead, alive}\}$ whether a given target pattern $p_{\text{target}}$ is ever reached, semi-decidable? Is it decidable?

https://en.wikipedia.org/wiki/Conway%27s_Game_of_Life
https://www.youtube.com/watch?v=vGWGeund3eA