Exercise 1: $\mathcal{O}$-Notation Formal Proofs

The set $\mathcal{O}(f)$ contains all functions that are asymptotically not growing faster than the function $f$ (when additive or multiplicative constants are neglected). That is:

$$g \in \mathcal{O}(f) \iff \exists c \geq 0, \exists M \in \mathbb{N}, \forall n \geq M : g(n) \leq c \cdot f(n)$$

For the following pairs of functions, check whether $f \in \mathcal{O}(g)$ or $g \in \mathcal{O}(f)$ or both.

(a) $f(n) = n$, $g(n) = n^2$

(b) $f(n) = 2^n$, $g(n) = 3^n$

(c) $f(n) = \log_2(n!)$, $g(n) = n \log_2 n$ \hspace{1cm} Hint: $n! := \prod_{i=1}^{n} i \geq (n/2)^{n/2}$
Is the following problem an optimization or a decision problem? Transform it into the respective other problem type.

**DominatingSet:**
- A *dominating set* of graph $G = (V, E)$ is a subset $D \subseteq V$, s.t. for every vertex $v \in V$: $v \in D$ or $v$ adjacent to a node $u \in D$.
- **Input:** Encoding $\langle G, k \rangle$ of an undirected, unweighted, simple graph $G = (V, E)$ and $k \in \mathbb{N}$.
- **Question:** Is there a dominating set with at most $k$ nodes?
Is the following problem an optimization or a decision problem? Transform it into the respective other problem type.

**VertexColoring:**

- A *vertex coloring* of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, \ldots, k\}$ such that $c(u) = c(v) \Rightarrow \{u, v\} \notin E$.
- **Input:** Encoding $\langle G \rangle$ of an undirected, unweighted, simple graph $G = (V, E)$.
- **Question:** What is the smallest $k$ for which a valid vertex coloring $c$ of $G$ exists?
Exercise 3: The class $\mathcal{P}$

$\mathcal{P}$ is the set of languages that can be decided by a TM (algorithm) whose run-time can be bounded from above by $p(n)$, where $p$ is a polynomial and $n$ the size of the respective input (problem instance).

Show that that the following languages are in $\mathcal{P}$.

(a) $L := \{w \in \{0, 1\}^* \mid w \text{ has even length}\}$
(b) Any regular Language.
(c) 1-DOMINATINGSET := $\{\langle G \rangle \mid G \text{ has dominating set of size } \leq 1\}$
(d) 2-VERTEXCOLORING := $\{\langle G \rangle \mid G \text{ has vertex coloring with } \leq 2 \text{ col.}\}$