Theoretical Computer Science - Bridging Course

Tutorial 08

Albert-Ludwigs-Universität Freiburg

Philipp Schneider

Algorithms and Complexity - Professor Dr. Fabian Kuhn

Exercise 1: The class *NP*

NP is the set of languages that can be decided by a **non-deterministic** Turing machine whose run-time can be bounded by $p(n)$, where p is a polynomial and n the size of the respective input.

Show that the following problems are in *NP*

- CLIQUE = $\{\langle G,k\rangle \mid$ Graph G has complete subgraph with k nodes$\}$
- VERTEXCOVER := $\{\langle G,k\rangle \mid$ Graph G has *vertex cover* of size $\leq k\}$
- 3-SAT = $\{\langle \phi\rangle \mid$ bool. formula ϕ in 3-CNF has assignment of variables s.t. ϕ evaluates to TRUE.$\}$

*Remark: A **vertex cover** is a subset $V' \subseteq V$ of nodes of $G = (V,E)$ such that every edge of G is adjacent to a node in the subset.*

*ϕ is in 3-CNF if it is of the form $C_1 \wedge \ldots \wedge C_m$, where $C_i = L_{i,1} \vee L_{i,2} \vee L_{i,3}$ are clauses of **at most** three **literals** $L_{i,j} = x_k$ or $L_{i,j} = \overline{x}_k$ of negated or non-negated **variables** $x_1, \ldots, x_n$ of ϕ.*

Exercise 2: Decision and Optimization Problems

Is the following problem an optimization or a decision problem? Transform it into the respective other problem type.

DOMINATINGSET:

- A *dominating set* of graph $G = (V, E)$ is a subset $D \subseteq V$, s.t. for every vertex $v \in V: v \in D$ or v adjacent to a node $u \in D$.
- **Input**: Encoding $\langle G, k \rangle$ of an undirected, unweighted, simple graph $G = (V, E)$ and $k \in \mathbb{N}$.
- **Question**: Is there a dominating set with at most k nodes?

Exercise 2: Decision and Optimization Problems

Is the following problem an optimization or a decision problem? Transform it into the respective other problem type.

VERTEXCOLORING:

- A *vertex coloring* of a graph $G = (V,E)$ is a mapping $c : V \to \{1,\dots,k\}$ such that $c(u) = c(v) \Rightarrow \{u,v\} \notin E$.
- **Input**: Encoding $\langle G \rangle$ of an undirected, unweighted, simple graph $G = (V,E)$.
- **Question**: What is the smallest k for which a valid vertex coloring c of G exists?

Exercise 3: The class *NPC*

Show that

$\textsc{IndependentSet} := \{\langle G, k\rangle \mid \text{graph } G \text{ has an } \textbf{independent set} \text{ of size } k\}.$

Is *NP*-complete. Use that

$$\textsc{Clique} := \{\langle G, k\rangle \mid \text{graph } G \text{ has a } \textbf{clique} \text{ of size } k\}.$$

is *NP*-complete.

*Remark: An **independent set** of size k of $G = (V, E)$ is a subset $I \subseteq V$ of nodes, such that $|I| = k$ and $\{u, v\} \notin E$ for all $u, v \in I, u \neq v$.*
*A **clique** of size k of $G = (V, E)$ is a subset $C \subseteq V$ of nodes, such that $|C| = k$ and $\{u, v\} \in E$ for all $u, v \in C, u \neq v$.*

Exercise 4: The class *NPC*

Show that

$$\textsc{DominatingSet} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has } \textbf{dominating set} \text{ of size} \leq k\}$$

Is *NP*-complete. Use that

$$\textsc{VertexCover} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has a } \textbf{vertex cover} \text{ of size} \leq k\}$$

is *NP*-complete.

Exercise 5: *NP*-hard Problems

Show that the Halting Problem H is NP-hard. Use that

$$\text{SAT} = \{\langle\phi\rangle \mid \text{bool. formula } \phi \text{ has assignment of variables s.t. } \phi \text{ is TRUE.}\}$$

is NP-hard.

Hint: For any boolean formula ϕ give an algorithm A that stops on input ϕ if and only if ϕ is satisfiable.