Theoretical Computer Science - Bridging Course

Tutorial 09

Albert-Ludwigs-Universität Freiburg

Philipp Schneider

Algorithms and Complexity - Professor Dr. Fabian Kuhn

Exam Notification

The exam will take place on the **17th of August at 10:00 am**. It will take **120 min**. The exam will be an **open-book exam**, which means you are allowed to bring any printed or written material. **Electronic equipment is not allowed**!

We recommend you to write a summary of the topics covered in the lecture. This has two advantages:

1. You will see the big picture and also learn the details (if your summary is well crafted and if you do it by yourself).
2. You can bring it to the exam in case you can't rememeber some definition (this is way more handy than a book which you have never worked with before).

Exercise 1: Propositional Logic

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An **interpretation** $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ. We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a **model** for φ.

Which of these formulae are **satisfiable** (have a model), which are **unsatisfiable** (have no model) and which are **tautologies** (all interpretations are models)?

(a) $\varphi_1 = (p \to q) \to r$

(b) $\varphi_2 = (p \wedge \neg p) \leftrightarrow (q \vee r)$

(c) $\varphi_3 = ((p \to q) \wedge (q \to r)) \to (p \to r)$

(d) $\varphi_4 = p \vee (p \wedge q) \not\leftrightarrow p$

Exercise 2: Logical Equivalency

Two logical formulae φ, ψ over a set of atoms Σ are logically equivalent ($\varphi \equiv \psi$) iff for all interpretations I of Σ the following holds

$$I \models \varphi \Longleftrightarrow I \models \psi.$$

With the above definition, show or disprove the following equivalencies (e.g. by making truth tables).

(a) $p \wedge (q \vee r) \stackrel{?}{\equiv} (p \wedge q) \vee (p \wedge r)$

(b) $\neg(p \rightarrow q) \stackrel{?}{\equiv} p \rightarrow \neg q$

(c) $\neg(p \wedge q) \stackrel{?}{\equiv} \neg p \vee \neg q$

Exercise 3: Logical Entailment

A **knowledge base** KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB **entails** a formula φ ($KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{\neg p \vee q, q \vee \neg r\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := \neg q$

(b) $\varphi_2 := (\neg p \wedge q) \vee \neg(\neg r \vee q)$

(c) $\varphi_3 := \neg(r \wedge \neg q) \vee (\neg q \rightarrow \neg p)$

Calculi

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.

Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

Exercise 4: Calculi

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \rightarrow \psi, \psi \rightarrow \chi}{\varphi \rightarrow \chi}, \frac{\neg\varphi \rightarrow \neg\psi}{\psi \rightarrow \varphi}, \frac{\varphi \rightarrow \psi, \psi \rightarrow \varphi}{\varphi \leftrightarrow \psi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{\varphi, \psi}{\varphi \wedge \psi}, \frac{\neg(\varphi \vee \psi)}{\neg\varphi \wedge \neg\psi}, \frac{\neg\neg\varphi}{\varphi}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \rightarrow r, \neg p \rightarrow \neg q, \neg q \rightarrow \neg r\} \vdash_{\mathbf{C_1}} p \leftrightarrow q$

(b) $\{\neg(\neg p \vee q), \neg q \rightarrow (r \vee s), (r \vee s) \rightarrow t\} \vdash_{\mathbf{C_2}} t \wedge p$