Theoretical Computer Science - Bridging Course

Tutorial 09

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Calculi

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.

Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

Calculi - Correctness and Completeness

A calculus **C** is called *correct* if for every knowledge base *KB* and formula φ the following holds

$$KB \vdash_{\mathbf{C}} \varphi \quad \Rightarrow \quad KB \models \varphi.$$

Calculus **C** is called *complete* if

$$KB \models \varphi \quad \Rightarrow \quad KB \vdash_{\mathbf{C}} \varphi.$$

Exercise 1: Calculi - Correctness and Completeness

Consider the following calculi

$$\mathbf{C_1}: \frac{\varphi \rightarrow \psi, \psi \rightarrow \chi}{\varphi \rightarrow \chi}, \quad \mathbf{C_2}: \frac{\neg(\varphi \vee \psi)}{\neg\varphi \vee \neg\psi}, \quad \mathbf{C_3}: \frac{\neg(\varphi \vee \psi)}{\neg\varphi \wedge \neg\psi}.$$

Refutation Complete - Calculi

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \iff KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ *is a formula that is unsatisfiable.*

Thus, in order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus **C** is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \implies KB \vdash_{\mathbf{C}} \bot.$$

Therefore, if we have a refutation-complete calculus **C**, it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ in order to prove $KB \models \varphi$.

Resolution Calculus

The *Resolution Calculus* **R** is correct and refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base *KB* is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$.

The Resolution Calculus consists only of the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty).

Exercise 2: Resolution Calculus

(a) We want to show $\{p \vee q, q \rightarrow (r \wedge s), (p \vee r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \vee q, q \rightarrow (r \wedge s), (p \vee r) \rightarrow u\} \models u$.

(c) Using resolution, show that $(p \wedge q) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg r) \vee (q \wedge r) \vee (p \wedge \neg q)$ is a tautology. *Hint:*
φ *tautology* $\Leftrightarrow \top \models \varphi \Leftrightarrow \neg\varphi \models \bot \Leftrightarrow \neg\varphi$ *unsatisfiable .*

(d) Assuming $P \neq NP$, argue why proving logical entailment via resolution as described above, can not be done in polynomial time. *Hint:* 3-*SAT* $\in$ *NPC.*

Predicate Logic

In *predicate logic* formulae φ are given with respect to a *signature* $\mathscr{S} = \langle \mathscr{V}, \mathscr{C}, \mathscr{F}, \mathscr{R} \rangle$. Its components are: The variable symbols $\mathscr{V}$, constant symbols $\mathscr{C}$, function symbols $\mathscr{F}$ and the set of relation symbols $\mathscr{R}$. Elements of $\mathscr{V}, \mathscr{C}, \mathscr{F}$ are used to formulate *terms*, while the relations in $\mathscr{R}$ compare terms with each other.

These components must be combined in a *well-formed* manner with logical connectives ($\wedge, \vee, \neg$, etc.), quantifiers ($\forall, \exists$), and the '$\doteq$' symbol which is a relation that represents equality of terms.

Exercise 3: Predicate Logic - Construct Formulae

Let $\mathscr{S} = \langle \{x\}, \emptyset, \emptyset, \{P, Q, R, S\} \rangle$ be a signature. Express each of the following statements as first order formula over $\mathscr{S}$. Use $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ as statements 'x is a duck', 'x is one of my poultry', 'x is an officer', and 'x is willing to waltz'.

(a) No ducks are willing to waltz.

(b) No officers ever decline to waltz.

(c) All my poultry are ducks.

(d) My poultry are not officers.

Predicate Logic - Interpretations

As it was the case in propositional logic, we require *interpretations* in order to evaluate first order formulae to true or false. An interpretation $I = \langle \mathscr{D}, \cdot^I \rangle$ has a *domain* $\mathscr{D}$ that represents the set of all values that variables can assume.

Furthermore an interpretation has a mapping $\cdot^I$ which assigns constant symbols $\mathbf{c} \in \mathscr{C}$ a fixed value $\mathbf{c}^I \in \mathscr{D}$ from the domain. A function symbol $f \in \mathscr{F}$ is assigned an explicit function $f^I : \mathscr{D}^k \to \mathscr{D}$. A relation symbol $R \in \mathscr{R}$ is assigned an explicit relation $R^I \subseteq \mathscr{D}^k$. The parameter k is called *arity*.

Exercise 4: Predicate Logic - Interpretations

Evaluate the given formulae with the given interpretations. Make clear why or why not an interpretation is a model for the formula.

(a) $\varphi_1 := \forall x \exists y\, f(y) \doteq x$,
$I_1 := \langle \mathbb{Z}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{Q}, \cdot^{I_2} \rangle$ where $f^{I_1}(a) := f^{I_2}(a) := 2 \cdot a$.

(b) $\varphi_2 := \forall x \exists y\, f(y,y) \doteq x$,
$I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{C}, \cdot^{I_2} \rangle$ where $f^{I_1}(a,b) := f^{I_2}(a,b) := a \cdot b$.

(c) $\varphi_3 := (\forall x\, f(x,z) \doteq x) \land (\forall x \exists y\, f(x,y) \doteq z)$,
$I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{R}, \cdot^{I_2} \rangle$ where
$f^{I_1}(a,b) := a + b, f^{I_2}(a,b) := a \cdot b$.

(d) $\varphi_4 := \forall x \forall y \forall z\, f(x, f(y,z)) \doteq f(f(x,y),z)$. Give a model and an interpretation that is no model.

Predicate Logic - Free variables

If a formula φ has *free variables* (variables that are not bound by a quantifier: $\forall, \exists$) then I requires an additional variable *assignment function* $\alpha : \mathscr{V} \rightarrow \mathscr{D}$ assigning each free variable a value from the domain. An interpretation I is called a model of a first order formula φ, if an assignment function α *exists* (!) such that $\varphi^{I,\alpha}$ evaluates to true.

Exercise 5: Predicate Logic - Entailment

Let φ, ψ be first order formulae over signature $\mathscr{S}$. Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of φ is also a model for ψ. We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base KB* is a set of formulae. A model of *KB* is model for all formulae in *KB*. We write $KB \models \varphi$ if all models of *KB* are models of φ. Show or disprove the following entailments.

(a) $(\forall x \forall y\, f(x,y) \doteq f(y,x)) \land (\forall x \exists y\, f(x,y) \doteq \mathbf{c}) \models \forall x \exists y\, f(y,x) \doteq \mathbf{c}$.

(b) $(\exists x\, R(x)) \land (\exists x\, P(x)) \equiv \exists x\, R(x) \land P(x)$.