Theoretical Computer Science - Bridging Course

Tutorial 09

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Pumping Lemma

Let Σ be an alphabet. Consider the *Pumping Lemma* in the following notation:

$$L \subseteq \Sigma^* \text{ regular} \Longrightarrow \exists p \in \mathbb{N} \ \ \forall s \in \{w \in L \mid |w| \geq p\}$$

$$\exists x, y, z \in \Sigma^* \text{ such that } s = xyz \ \text{ and}$$

$$(1) \ |xy| \leq p \ \text{ and}$$

$$(2) \ |y| > 0 \ \text{ and}$$

$$(3) \ \forall i \in \mathbb{N}_0 \ \ xy^iz \in L$$

Exercise 1: Pumping Lemma

(a) Show that $L_1 := \{a^n b^m c^n \mid n \geq 0\}$ is not regular.

(b) Show that $L_2 := \{www \mid w \in \{a,b\}^*\}$ is not regular.

(c) Show that $L_3 := \{a^{2^n} \mid n \in \mathbb{N}_0\}$ is not regular.

(d) Show that **any finite** language is regular. Does this result conflict with the Pumping Lemma?

(e) Give a Venn-Diagram showing the relation between the set of all languages over Σ, the set of regular languages over Σ and the set of languages over Σ for which the right hand side of the Pumping Lemma holds.

Pumping Lemma for Context-Free Languages

Let Σ be an alphabet. Consider the *Pumping Lemma* in the following notation:

$$
\begin{aligned}
L \subseteq \Sigma^* \text{ context-free} \Longrightarrow \exists p \in \mathbb{N}\ \ \forall s \in \{w \in L \mid |w| \geq p\}\\
\exists u, v, w, x, y \in \Sigma^* \text{ such that } s = uvwxy \ \text{ and}\\
(1)\ |vwx| \leq p \ \text{ and}\\
(2)\ |vx| > 0 \ \text{ and}\\
(3)\ \forall i \in \mathbb{N}_0\ \ uv^iwx^iy \in L
\end{aligned}
$$

Exercise 2: Pumping Lemma for Context-Free Languages

(a) Let $L_2 = \{a^n b^n c^n \mid n \in \mathbb{N}_0\}$.
Prove that L is not a context-free language.

(b) Let $L_1 = \{w \in \{1,2,3,4\}^* \mid |w|_1 = |w|_2, |w|_3 = |w|_4\}$.
Here, $|w|_n$ denotes the number of occurrences of n in w.
Show that L is not context-free.

Exercise 3: (Semi-)Decidability

(a) Show that the following problem is decidable
$\text{SAT} = \{\varphi \mid \text{propositional formula } \varphi \text{ can be satisfied}\}$.

(b) Show that the following problem is decidable
$\textsc{Clique} = \{\langle G, k\rangle \mid \text{Graph } G \text{ has a clique of size } k\}$.

(c) Show that the Halting problem
$H := \{\langle M, s\rangle \mid \text{Turing machine } M \textit{ halts} \text{ on input } s\}$
is semi-decidable[1].

(d) Show that the Halting problem H is undecidable.
Hint: You may use that we know that
$U := \{\langle M, s\rangle \mid \text{Turing machine } M \textit{ accepts} \text{ input } s\}$ *is an undecidable language from the lecture.*

[1] Our definition of the halting problem deviates from the one on the lecture slides, but is also very common.