Albert-Ludwigs-Universität
Institut für Informatik
Prof. Dr. F. Kuhn
Juho Hirvonen

August 15, 2018

Network Algorithms, Summer Term 2018
Problem Set 11 – Sample Solution

Counting with Asynchronous Wake-up (Start)

1. Every node that wakes up starts the algorithm as if it was the only node that woke up by itself, but includes its identifier and the current round number into all messages. Every node that is woken up by a message will append the received identifier and the (current) local round number of the respective execution to its messages. If a node that is already awake receives a message with a larger round number or the same round number and a smaller identifier, it "forgets" about everything that happend before and acts as if woken up by this message. If the round number is smaller or the round number equals its own, but the identifier is larger, it simply drops the message. If both are the same, it clearly belongs to the execution initiated by the same node causing this node to wake up and is processed normally.

   In the end, only the execution started by the node with smallest identifier among the nodes that woke up first will "survive" and yield the same output as if executed in an environment where no other nodes wake up by themselves.[1]

2. Consider the following algorithm.

   Algorithm. The algorithm consists of all (awake) nodes broadcasting all identifiers they know for $2k$ rounds. Then, after $2k$ rounds, u (and only u) outputs '$k \geq n$' if it knows at most k identifiers and '$k < n$' otherwise.

   Correctness. We split the proof of correctness in two parts. At first we show that u outputs '$k \geq n$' if $k \geq n$ holds and then we show that u outputs '$k < n$' if $k < n$ holds:

   Suppose $k \geq n$ holds. The node u can collect at most n identifiers, in particular it learns at most k identifiers. Hence the algorithm returns '$k \geq n$'. The algorithm works correctly for this case.

   Now, suppose $k < n$:

   Claim 1. *In round $2s$ where $s \in \{0, \dots, k\}$, any subset of $r + 1 - s$ awake nodes such that $r \in \{s, \dots, n-1+s\}$, all together will know (in total) $\min\{r+1, n\}$ identifiers.*

   Proof. We prove the claim by an induction on s combined with a nested induction on r.

   Base Case (for s). For $s = 0$ and all $r \in \{0, \dots, n-1\}$, any subset of $r+1$ awake nodes know $\min\{r+1, n\}$ identifiers since each of these nodes knows its own identifier right from the start.

   Induction Hypothesis (for s). Assume the claim holds for some $s \in \{0, \dots, k-1\}$ and all $r \in \{s, \dots, n-1+s\}$.

   Induction Step (for s). Here we need to show that the claim also holds for $s' = s+1$ and $r' \in \{s', \dots, n+s'-1\}$. The proof is established using an induction on r'. As *base case* (for

[1] Note that any protocol needs to obtain information from all nodes, thus nodes cannot terminate too early and "miss" the execution initiated by this node.

r') we show that the claim holds for s' and $r' = s'$. Then we show that it also holds for s' and $r' + 1$ (as *induction step* (for r')) while we assume the claim holds for s' and r' where $r' \in \{s', \dots, n + s' - 2\}$ (as *induction hypothesis* (for r')).

Base Case (for r'). For s' and $r' = s' \leq k < n$ we need to consider only a single node v since any subset of $r' + 1 - s'$ awake nodes has only one node. Two cases need to be considered:

(a) If v wakes up in round $2s+1$ or $2s+2$, it woke up due to a message from some node u. Using the induction hypothesis with s and $r = s$ this message contains (at least) $r+1 = s+1 = r'$ identifiers which must be different from v's identifier as v did not send any messages before. Combining these r' identifiers with v's own identifier v knows (at least) $r' + 1$ identifiers.

(b) If v was awake in round $2s$, the 2-interval connectivity implies that there is at least one edge $\{v, w\}$ from v to some node w that is in $E(2s + 1) \cap E(2s + 2)$.

   i. If w is still asleep at the end of round $2s$, it will wake up in round $2s + 1$ by a message from v and broadcast its identifier in round $2s + 2$. Since v knows at least $r + 1 = s + 1$ identifiers by induction hypothesis (on s for s and $r = s$) at the end of round $2s$, at the end of round $2s + 2$ it knows $r + 2 = s + 2 = r' + 1$ identifiers including the identifier of w.

   ii. If w is awake at the end of round $2s$, we use the induction hypothesis (on s) for s and $r = s + 1 \leq k < n$ where any subset of two nodes are awake, implying that v and w together know at least $s + 2 = r' + 1$ identifiers in round $2s$, which they will exchange in round $2s + 1$.

Thus, the claim holds for $s' = s + 1$ and $r' = s'$ and the base case (on r') follows.

Induction Step (for r'). We show that the claim holds for s' and $r'+1$ where $r' \in \{s', \dots, n + s' - 2\}$. Choose any set of nodes $S = \{v_1, \dots, v_{r'+1-s'}\}$ that are awake in round $2s + 2$. Again two cases are considered and the reasoning is almost exactly the same as in the base case for r':

(a) If (at least) one of the nodes has been asleep at the end of round $2s$, it will certainly contribute a new identifier by round $2s+2$. On the other hand, the remaining $r' - s'$ awake nodes (in round $2s + 2 = 2s'$) using the induction hypothesis (on r') for s' and $r' - 1$, together know in total r' different identifiers. Therefore all the nodes in S know $r' + 1$ identifiers in round $2s + 2$.

(b) If all the nodes are awake in round $2s$, there must be at least one edge from an (awake) node in S to some node w that that is in $E(2s + 1) \cap E(2s + 2)$.

   i. If w is asleep in round $2s$, implying that it wakes up in round $2s+1$ and a node in S will learn its identifier which previously was unknown to all nodes in S. Using hypothesis (on s) for $s' - 1 = s$ and $r' - 1$ we can see that in round $2s$, the $r' - s' + 1$ nodes in S knew r' identifiers in which together with identifier of w they know at least $r' + 1$ identifiers at the end of round $2s + 2$.

   ii. If all nodes from $S \cup \{w\}$ were awake in round $2s$, we can apply the hypothesis (on s) for $s' - 1 = s$ and r' to see that in round $2s$, $S \cup \{w\}$ knew $r' + 1$ identifiers. In round $2s + 1$, a node in S will receive all identifiers known to w, hence the same is true for the set S in round $2s + 2$.

Therefore the statement of the claim is proved by induction for all possible values of s and r. □

In particular, setting $s = r = k$, node u (that is, the set containing only the element u) will know $\min\{k + 1, n\}$ elements using Claim 1. We have $\min\{k + 1, n\} = k + 1$ because $k < n$ and thus node u learns more than k ids; hence u will decide correctly and returns '$k < n$'.

3. Starting from $k = 1$, u runs the algorithm from part **2** repeatedly, in each step doubling k. Note that in the final run, u will learn about all $n \leq k$ identifiers, thus it can determine n exactly. Moreover, actually all nodes are awake and therefore aware of the fact that $k \geq n$ and they know

all identifiers. Hence the algorithm may terminate and output n at all nodes without need for further intervention of u.

We conclude that the time complexity of the algorithm is the sum of the number of rounds the individual runs of the algorithm from part **2** take, i.e.,

$$\sum_{i=0}^{\lceil \log n \rceil} 2 \cdot 2^i = 2\left(2^{\lceil \log n \rceil + 1} - 1\right) < 8n.$$