Exercise 1: Resolution Calculus

Considering each of the following cases, first convert the knowledge base ($KB_i$) and the formula ($\varphi_i$) to CNFs. Then, by resolution, show that the knowledge base entails the formula.

(a) $KB_1 := \{(x \land y) \rightarrow (z \lor w), \ y \rightarrow x, \ (z \land y) \rightarrow 0, \ y\}$

$\varphi_1 := w \land y$

(b) $KB_2 := \{\neg A \rightarrow B, \ B \rightarrow A, \ A \rightarrow (C \land D)\}$

$\varphi_2 := A \land C \land D$

Sample Solution

(a) $KB_1 = \{(\neg x \lor \neg y \lor z \lor w), (\neg y \lor x), (\neg z \lor \neg y), y\}$

$\varphi_1 = w \land y$

We add the negation of the formula to the knowledge base and try to reach a contradiction by applying the resolution rule.

$KB'_1 := \{(\neg x \lor \neg y \lor z \lor w), (\neg y \lor x), (\neg z \lor \neg y), y, (\neg w \lor \neg y)\}$

$\{\neg x \lor \neg y \lor z \lor w), \ (\neg y \lor x), \ (\neg z \lor \neg y), \ y, \ (\neg w \lor \neg y)\} \vdash R \{\neg x \lor z \lor w\}$

$\{\neg y \lor x\} \vdash R \{x\}$

$\{\neg x \lor z \lor w\}, \ x \vdash R \{(z \lor w)\}$

$\{\neg z \lor \neg y\} \vdash R \{\neg z\}$

$\{z \lor w\}, \ \neg z \vdash R \{w\}$

$\{\neg w \lor \neg y\}, \ y \vdash R \{\neg w\}$

$\{w, \ \neg w\} \vdash R []$

(b) $KB_2 = \{(A \lor B), (\neg B \lor A), (\neg A \lor C), (\neg A \lor D)\}$

$\varphi_2 = A \land C \land D$

We add the negation of the formula to the knowledge base and try to reach a contradiction by applying the resolution rule.
Exercise 2: Implication vs. Entailment (5 Points)

Show that \( P \models Q \iff (\text{True} \models P \rightarrow Q) \)

Sample Solution

Let \( T(P) \) and \( T(Q) \) be the set of models for \( P \) and \( Q \) respectively.

(a) \( \rightarrow \): Let us assume that \( P \models Q \). By the definition of entailment, we have \( T(P) \subseteq T(Q) \). Moreover, since there is no interpretation under which \( Q \) and \( \neg Q \) are both true, \( T(Q) \cap T(\neg Q) = \emptyset \). Therefore, it implies that \( T(P) \cap T(\neg Q) = \emptyset \). It means that there is no interpretation under which \( P \) is true while \( Q \) is false. Hence, under all interpretations, \( P \rightarrow Q \) is true. That is \( T(P \rightarrow Q) = T(\text{True}) \), and consequently \( T(\text{True}) \subseteq T(P \rightarrow Q) \). By the definition of entailment, it concludes \( \text{True} \models P \rightarrow Q \).

(b) \( \leftarrow \): Let us assume that \( \text{True} \models P \rightarrow Q \). By the definition of entailment, this means that \( P \rightarrow Q \) is true under all interpretations, and therefore there is no interpretation under which \( P \) is true and \( Q \) is false, i.e., \( T(P) \cap T(\neg Q) = \emptyset \). Therefore, \( T(P) \subseteq T(Q) \) and we can conclude that \( P \models Q \).

(a) and (b) together concludes the prove of the statement.

Exercise 3: Understanding First Order Logic (2+2+2 Points)

Consider the following first order logical formulae

\[
\varphi_1 := \forall x R(x, x) \\
\varphi_2 := \forall x \forall y R(x, y) \rightarrow (\exists z R(x, z) \land R(z, y)) \\
\varphi_3 := \exists x \exists y (\neg R(x, y) \land \neg R(y, x))
\]

where \( x, y \) are variable symbols and \( R \) is a binary predicate. Give an interpretation

(a) \( I_1 \) which is a model of \( \varphi_1 \wedge \varphi_2 \).

(b) \( I_2 \) which is no model of \( \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \).

(c) \( I_3 \) which is a model of \( \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \).

Sample Solution

(a) Pick \( I_1 := \langle \mathbb{R}, I_1 \rangle \) where \( R^{I_1}(x, y) \iff x \leq_R y \).

This is a model because \( \leq_R \) is reflexive, therefore fulfills \( \varphi_1 \). Moreover for every \( x, y \in \mathbb{R} \) with \( x \leq_R y \) we can choose \( z := x \), which fulfills \( x \leq_R z \land z \leq_R y \). Thus \( \varphi_2 \) is also satisfied.

(b) Pick \( I_2 := \langle \mathbb{R}, I \rangle \) where \( R^I(x, y) = \text{false} \).

This is not a model since it violates \( \varphi_1 \), e.g. \( R^I(5, 5) = \text{false} \).
(c) Take two disjoint copies of $\mathbb{R}$ and the standard $\leq_{\mathbb{R}}$ relation on each of them; if $x$ and $y$ are from different copies they are not related in $\mathbb{R}$. Formally let

$$I_3 := \langle \{(a, 1) \mid a \in \mathbb{R}\} \dot{\cup} \{(a, 2) \mid a \in \mathbb{R}\}, \cdot \rangle$$

where $R_{I_3}((a, g), (b, h)) \iff (g = h$ and $a \leq_{\mathbb{R}} b)$.

This is a model because $\leq_{\mathbb{R}}$ is reflexive, therefore $I_3$ fulfills $\varphi_1$. Furthermore for every two $x = (a, g)$ and $y = (b, h)$ with $R_{I_3}((a, g), (b, h))$, i.e., $g = h$, we can choose $z := (a, g)$ which fulfills $R_{I_3}((a, g), (a, g)) \land R_{I_3}((a, g), (b, h))$. Thus $\varphi_2$ is also satisfied. $\varphi_3$ is also satisfied, e.g., $(5, 1)$ and $(7, 2)$ are incomparable, i.e., we have neither $R_{I_3}((5, 1), (7, 2))$ nor $R_{I_3}((7, 2), (5, 1))$.

**Exercise 4: Truth Value**

(1+1+1 Points)

Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain (or universe) for the variables consists of:

(a) the positive real numbers,

(b) the integers,

(c) the nonzero real numbers.

**Sample Solution**

(a) This is false, since no matter how small a positive number $x$ we might choose, if we assume $y = \sqrt{x/2}$, then $x = 2y^2$, and it will not be true that $x \leq y^2$.

(b) This is true, because we can take $x = -1$ as an example.

(c) This is true, since we take $x = -1$. 

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