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Advanced Algorithms
Problem Set 1

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Exercise 1: Set Cover Integrality Gap

Consider some combinatorial optimization problem $\mathcal{P}$ that can be phrased as an integer linear program (ILP) and let $\mathcal{P}_f$ be the corresponding LP relaxation. The *integrality gap* of such a problem $\mathcal{P}$ is defined as the maximum possible ratio between the value of an optimal solution for $\mathcal{P}$ and the value of an optimal solution for the LP relaxation $\mathcal{P}_f$. The integrality gap is the best possible approximation guarantee one can hope for when using LP-based methods to design/analyze an approximation algorithm. The goal of this exercise is to bound the integrality gap of the set cover problem for the LP relaxation that we considered in the lecture.

(a) Show that for every integer $f \geq 2$, there exists an unweighted set cover instance $(E, \mathcal{S})$ with maximum element frequency f for which the integrality gap is equal to at least $f - \varepsilon$ for any (arbitrarily small) constant $\varepsilon > 0$.

(b) Show that there exists a constant $c > 0$ such that for every sufficiently large positive integer m, there exists an unweighted set cover instance $(E, \mathcal{S})$ with $|E| = m$ elements for which the integrality gap is at least $c \ln m$.

Exercise 2: Minimum Membership Set Cover

We now consider the minimum membership set cover problem, which is a variation of the set cover problem. As in the minimum set cover problem, we want to compute a set cover $\mathcal{C}$ of a given set system $(E, \mathcal{S})$ with $n = |E|$ elements. However, instead of minimizing the cardinaliy $|\mathcal{C}|$ of $\mathcal{C}$, we want to minimize the maximum number of times an element is covered by a set in $\mathcal{C}$. Formally, we want to find a set cover $\mathcal{C}$ that minimizes

$$\max_{e \in E} \left|\{S \in \mathcal{C} \,:\, e \in S\}\right|.$$

(a) Show that it is NP-hard to approximate the minimum membership set cover problem within a factor $(1-\varepsilon)\ln n$ for any constant $\varepsilon > 0$.

Hint: You can use that it is NP-hard to approximate the (unweighted) minimum set cover problem within a factor $(1-\varepsilon)\ln n$ *for any constant* $\varepsilon > 0$.

(b) Phrase the above problem as an integer linear program.

(c) Show that by solving an appropriate LP relaxation of your ILP, combined with randomized rounding, one can obtain $O(\log n)$-approximation of the minimum membership set cover problem with probability at least 1/2.

Exercise 3: Minimum Weighted Set Double Cover

Next, we consider a variation of the weighted set cover problem that we call the minimum weighted set double cover. Assume that we are given a set system $(E, \mathcal{S})$ such that each element $e \in E$ is contained in at least two different sets in $\mathcal{S}$. Further, each set $S \in \mathcal{S}$ is assigned a positive weight $w(S) > 0$. The goal now is to choose a collection $\mathcal{C} \subseteq \mathcal{S}$ of sets such that each element $e \in E$ is contained in at least 2 of the sets in $\mathcal{C}$. The total weight should be as small as possible.

(a) Phrase the above problem as an integer linear program.

(b) Let $n = |E|$ be the number of elements. Show that by solving an appropriate LP relaxation of your ILP, combined with randomized rounding, one can obtain $O(\log n)$-approximation of the minimum weighted set double cover problem with probability at least $1/2$.

Hint 1: You need to make sure that the variables of your LP only take values between 0 *and* 1*.*

Hint 2: Choose the probabilities for the rounding step sufficiently large so that the last step of the rounding algorithm from the lecture is not necessary.

(c*) Let Δ be the largest set size. Show that by slightly adapting the randomized rounding algorithm for the usual minimum weighted set cover problem from the lecture, you can obtain an approximation algorithm for the minimum weighted set double cover problem with expected approximation ratio at most $\ln(2\Delta) + 1$.

Hint: Note that this subproblem is significantly more challenging. Changing the LP such that the variables can only take values between 0 *and* 1 *also changes the structure of the dual LP, which makes the last step of the rounding algorithm of the lecture more tricky.*