University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
P. Schneider

Advanced Algorithms
Problem Set 2

Issued: Friday May 3, 2019

Exercise 1: Random Walk on a Line and on a d-dimensional Mesh

Consider a random walk on the integers. The walk starts at 0 and in each step, it either moves 1 to the right or 1 to the left, each with probability 1/2. That is, if $W(t)$ is the position of the walk after t steps, we have $W(0) = 0$ and $W(t+1) = W(t) \pm 1$. Show that during the first n steps, with probability at least $1 - 1/n$, the walk never ends further than $O(\sqrt{n \log n})$ from where it started, i.e., for all $t \leq n$, we have $|W(t)| = O(\sqrt{n \log n})$ with probability at least $1 - 1/n$.

(a) Express the position $W(t)$ of the walk after t steps as a sum of t independent random variables.

(b) Develop a Chernoff bound as in the lecture, to upper bound the probability that $W(t) \geq d$ for some $d \geq 0$.

Hint: In order to upper bound $\mathbb{P}\big(W(t) \geq d\big)$, *you can use that for all* $x \in \mathbb{R}$: $(e^x + e^{-x})/2 \leq e^{x^2/2}$.

(c) Use the derived bound to show the claim about $|W(t)|$ for $t \leq n$.

Remark: If you did not succeed in (b), you can also use the Chernoff bound from the lecture to prove that $|W(t)| = O(\sqrt{n \log n})$ *with probability at least* $1 - 1/n$ *for all* $t \leq n$.

(d) Let us now consider a random walk on the d-dimensional mesh. The walk starts at the origin $(0, \ldots, 0)$ and in each step, it picks a uniformly random one of the d dimensions and walks one step in the positive or in the negative direction in that dimension (each with probability 1/2). Show that after n steps, the Euclidean distance to the origin is at most $O(\sqrt{n \log n})$ with probability at least $1 - 1/n$. Note that your argument should work even if $d \in o(n/\ln n)$ is super-constant.

Exercise 2: Graph Connectivity

Let $G = (V, E)$ be a graph with n nodes and edge connectivity[1] $\lambda \geq \frac{16 \ln n}{\varepsilon^2}$ (where $0 < \varepsilon < 1$). Now every edge of G is removed with probability $\frac{1}{2}$. We want to show that the resulting graph $G' = (V, E')$ has connectivity $\lambda' \geq \frac{\lambda}{2}(1-\varepsilon)$ with probability at least $1 - \frac{1}{n}$. This exercise will guide you to this result.

Remark: If you don't succeed in a step you can use the result as a black box for the next step.

(a) Assume you have a cut of G with size $k \geq \lambda$. Show that the probability that the same cut in G' has size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ is at most $e^{-\frac{\varepsilon^2 k}{4}}$.

(b) Let $k \geq \lambda$ be fixed. Show that the probability that at least one cut of G with size k becomes a cut of size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ in G' is at most $e^{-\frac{\varepsilon^2 k}{8}}$.

Hint: You can use that for every $\alpha \geq 1$, *the number of cuts of size at most* $\alpha\lambda$ *is at most* $n^{2\alpha}$.

(c) Show that for large n the probability that at least one cut of G with *any* size $k \geq \lambda$ becomes a cut of size *strictly smaller* than $\frac{k}{2}(1-\varepsilon)$ in G', is at most $\frac{1}{n}$.

Hint: Use another union bound.

[1] The connectivity of a graph is the size of the smallest cut $(S, V \setminus S)$ in G.