Chapter 1
Set Cover

Advanced Algorithms

SS 2019

Fabian Kuhn

Approximation Algorithms

An approximation algorithm is an algorithm that computes a solution for an optimization problem with an objective value that is provably within a bounded factor of the optimal objective value.

Formally:

- $\mathrm{OPT} \geq 0$: optimal objective value
  $\mathrm{ALG} \geq 0$: objective value achieved by the algorithm

- **Approximation Ratio α:**

$$\textbf{Minimization}: \alpha := \max_{\text{input instances}} \frac{\mathrm{ALG}}{\mathrm{OPT}}$$

$$\textbf{Maximization}: \alpha := \min_{\text{input instances}} \frac{\mathrm{ALG}}{\mathrm{OPT}}$$

Set Cover

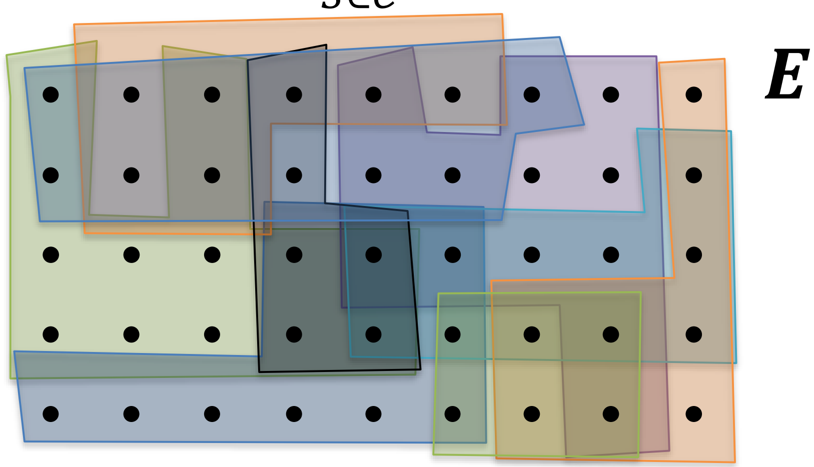
Input: A set of elements E and a collection $\mathcal{S}$ of subsets E, i.e., $\mathcal{S} \subseteq 2^E$

- such that $\bigcup_{S \in \mathcal{S}} S = E$, $|E| = n$
- Maximum set size $\Delta := \max_{S \in \mathcal{S}} |S|$
- Maximum element frequency $f := \max_{e \in E} |\{S \in \mathcal{S} : e \in S\}|$

Set Cover: A set cover $\mathcal{C}$ of $(E, \mathcal{S})$ is a subset of the sets $\mathcal{S}$ which covers E:

$$\bigcup_{S \in \mathcal{C}} S = E$$

Example:



Minimum (Weighted) Set Cover

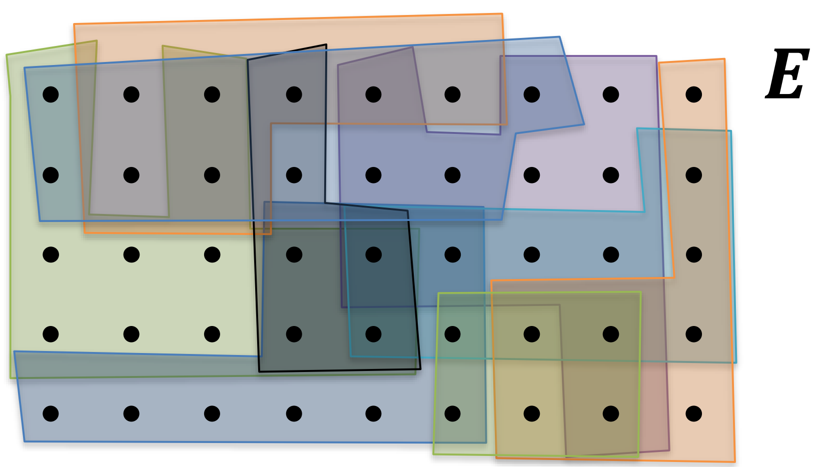
Minimum Set Cover:

- **Goal:** Find a set cover $\mathcal{C}$ of smallest possible size
  - i.e., over E with as few sets as possible

Minimum Weighted Set Cover:

- Each set $S \in \mathcal{S}$ has a weight $w(S) > 0$
- **Goal:** Find a set cover $\mathcal{C}$ of minimum weight

Example:

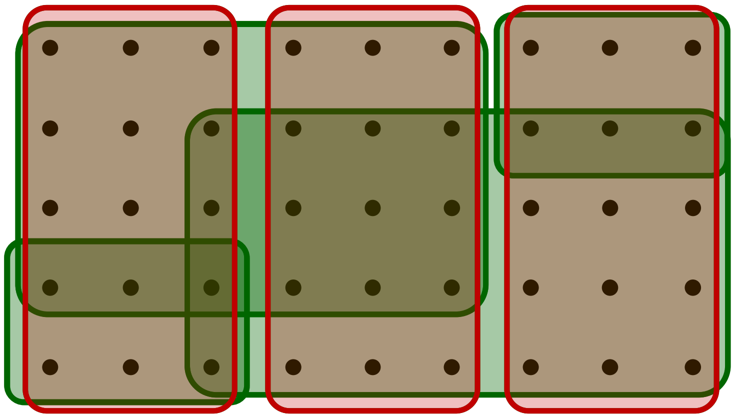


Minimum Set Cover: Greedy Algorithm

Greedy Set Cover Algorithm:

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ to $\mathcal{C}$ s.t. S covers as many uncovered elements as possible

Example:



Weighted Set Cover: Greedy Algorithm

Greedy Weighted Set Cover Algorithm:

- Start with $\mathcal{C} = \emptyset$
- Price-per-element ratio of $S \in \mathcal{S} \setminus \mathcal{C}$:

$$\mathrm{ppe}(S) := \frac{w(S)}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ with minimum $\mathrm{ppe}(S)$

Analysis of Greedy Algorithm:

- Assign a $\mathrm{price}(e)$ to each element $e \in E$:
  (price-per-element when covering the element)
- If covering e with set S and partial cover is $\mathcal{C}$ before adding S:

$$\mathrm{price}(e) = \mathrm{ppe}(S)$$

Weighted Set Cover: Greedy Algorithm

Lemma: Consider a set $S = \{e_1, e_2, \dots, e_k\} \in \mathcal{S}$ and assume that the elements are covered in the order $e_1, e_2, \dots, e_k$ by the greedy algorithm (ties broken arbitrarily).

Then, the price of element e_i is at most $\mathrm{price}(e_i) \leq \frac{w(S)}{k-i+1}$

Weighted Set Cover: Greedy Algorithm

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Corollary: The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{e \in S} \mathrm{price}(e) \leq w(S) \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

Weighted Set Cover: Greedy Algorithm

Corollary: The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{e \in S} p(e) \leq w(S) \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

Theorem: The approximation ratio of the greedy minimum (weighted) set cover algorithm is at most $\boldsymbol{H_\Delta \leq 1 + \ln \Delta}$, where s is the cardinality of the largest set ($\Delta = \max_{S \in \mathcal{S}} |S|$).

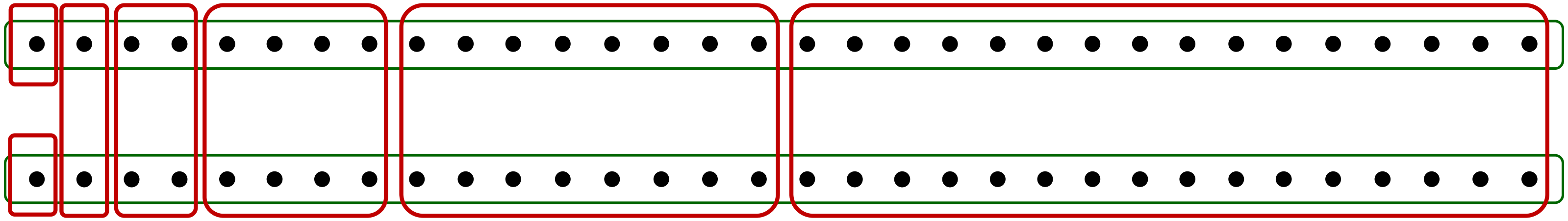
Set Cover Greedy Algorithm

Can we improve this analysis?

No! Even for the unweighted minimum set cover problem, the approximation ratio of the greedy algorithm is $\geq \big(1 - o(1)\big) \cdot \ln \Delta$.

- if Δ is the size of the largest set... (Δ can be linear in n)

Let's show that the approximation ratio is at least $\Omega(\log n)$...



$\mathbf{OPT = 2}$

$\mathbf{GREEDY \geq \log_2 \mathit{n}}$

Set Cover: Better Algorithm?

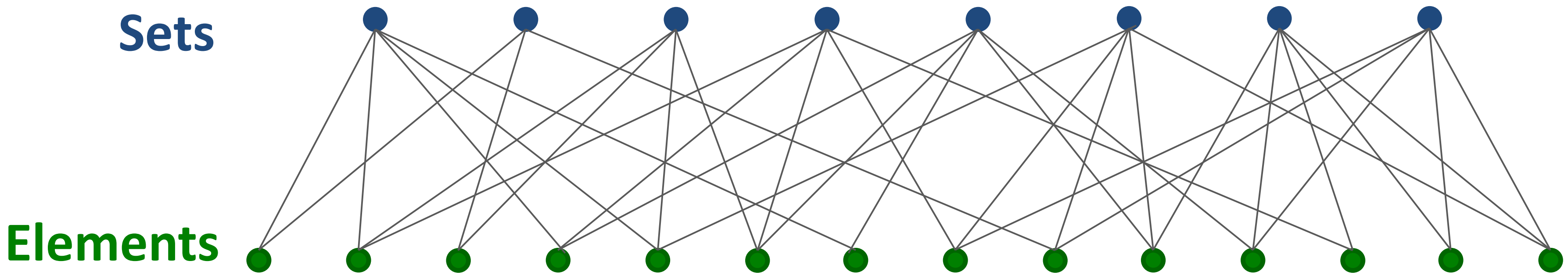
An approximation ratio of $\ln n$ seems not spectacular...

Can we improve the approximation ratio?

No: In a series of work, Lund and Yannakakis (1994), Feige (1998), and Moshkovitz (2015) showed that it is NP-hard to approximate minimum set cover by a factor $(1-\varepsilon) \cdot \ln n$ for any constant $\varepsilon > 0$.

- Proof is based on the so-called PCP theorem
  - PCP theorem is one of the main (relatively) recent advancements in theoretical computer science and the major tool to prove approximation hardness lower bounds
  - Shows that every language in NP has certificates of polynomial length that can be checked by a randomized algorithm by only querying a constant number of bits (for any constant error probability)

Special Case: Small f



Formulation as Minimum Hypergraph Vertex Cover

- Hypergraph $H = (V, E)$, $E \in 2^H$ are the hyperedges

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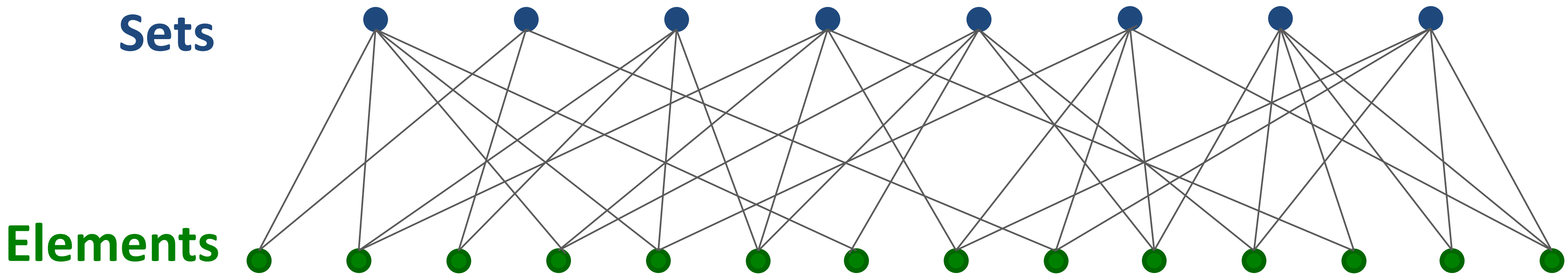
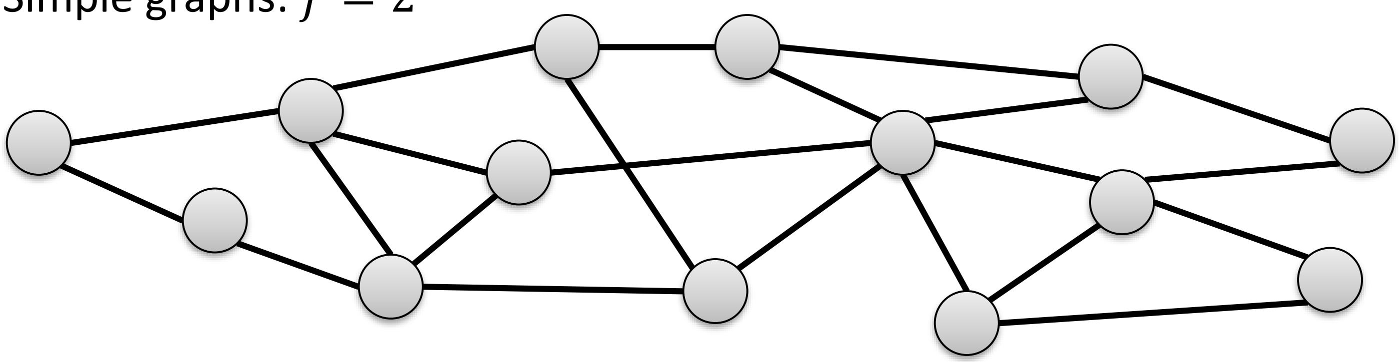


Formulation as Minimum Hypergraph Vertex Cover

- Hypergraph $H = (V, E)$, $E \in 2^H$ are the hyperedges
- Vertex cover: $S \in V$ s.t. $\forall e \in E : S \cap e \neq \emptyset$
  - equivalent to set cover (V: sets, E: elements)
  - Max. frequency f = max. hyperedge size = rank of H
  - Simple graphs: $f = 2$



Vertex Cover vs Matching

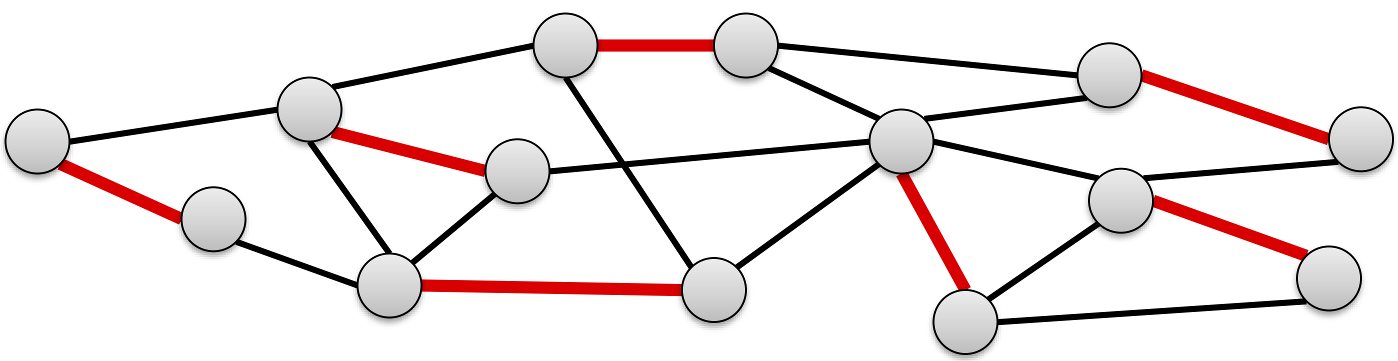
Matching of a hypergraph $\boldsymbol{H = (V, E)}$

- A disjoint set of edges $M \subseteq E$

Lemma: Given a hypergraph $H = (V, E)$, for every matching $M \subseteq E$ and every vertex cover $S \subseteq V$, we have $|M| \leq |S|$.

Proof:

- S is a vertex cover $\Longrightarrow \forall e \in M, \exists v_e \in e \cap S$
- M is a matching $\Longrightarrow v_{e_1} \neq v_{e_2}$ for $e_1 \neq e_2$ (e_1 & e_2 are disjoint)



Matching Approximation of Vertex Cover

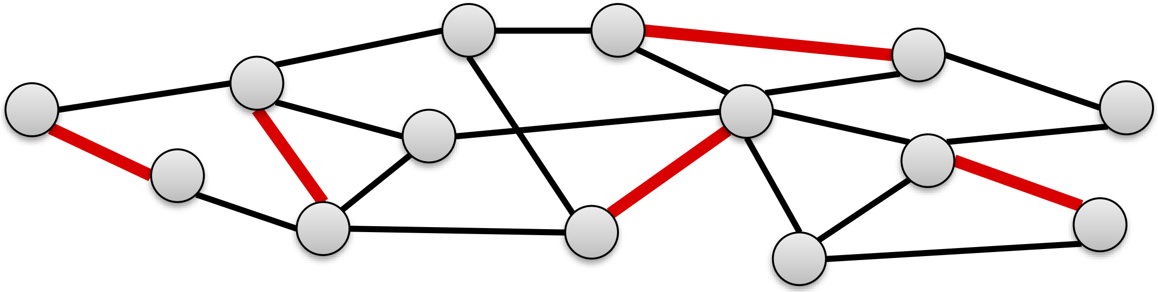
Vertex Cover Approximation Algorithm

- Let $H = (V, E)$ be a hypergraph of rank $\leq f$
- Compute a maximal matching M of H
- Define vertex cover S as $S \coloneqq \bigcup_{e \in M} e$

Theorem: The above algorithm computes an f-approximation of the (unweighted) minimum vertex cover problem in H.

Proof:

- M maximal $\Longrightarrow S$ is a vertex cover
  - $\forall \{v_1, \dots, v_k\} \in E$, at least one of vertices $v_1, \dots, v_k$ is matched



- We have $|S| = \sum_{e \in M} |e| \leq f \cdot |M|$ and $|M| \leq |S^*|$

$$\Longrightarrow |S| \leq f \cdot |S^*|$$

Linear Programming-Based Formulation

Linear Program (LP)

- (Continuous) optimization of a linear objective function subject to linear constraints

$$\begin{aligned} &\min \boldsymbol{c}^T \boldsymbol{x} \\ &\text{s.t. } A\boldsymbol{x} \geq \boldsymbol{b} \\ &\qquad\ \ \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

LP Duality

- Every LP has a dual LP

Linear Program

$$\min \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t. } A\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual Linear Program

$$\max \boldsymbol{b}^T\boldsymbol{y}$$
$$\text{s.t. } A^T\boldsymbol{y} \leq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

- **Weak duality:** For feasible solutions $\boldsymbol{x}$ and $\boldsymbol{y}$: $\boldsymbol{b}^T\boldsymbol{y} \leq \boldsymbol{c}^T\boldsymbol{x}$

- **Strong duality:** For optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$: $\boldsymbol{b}^T\boldsymbol{y}^* = \boldsymbol{c}^T\boldsymbol{x}^*$

LP-Based Approximation Algorithms

Important Technique to Design Approximation Algorithms

- LPs can be solved optimally in polynomial time
  - Using interior-point methods [Khachiyan '79], [Karmarkar '84]
- Many combinatorial optimization problems can be phrased as an integer linear program (ILPs):
  - LP with additional constraint that variables have to take integer values

Basic idea of many approximation algorithms:

1. Formulate given problem as an ILP
2. Relax integer constraints to get an LP
   - known as the LP relaxation of the given ILP
3. Solve the LP
4. Convert (fractional) LP solution to an integer solution
   - typically the hard part …

Minimum Set Cover as an ILP

Given: set system $(X, \mathcal{S})$ and weight $w(S) > 0$ for all S

Fractional Set Cover

- LP relaxation gives variables $x_S \geq 0$ for each $S \in \mathcal{S}$, s.t.

$$\forall e \in E : \sum_{S:e\in S} x_S \geq 1$$

  and s.t. $\sum_{S\in\mathcal{S}} x_S \cdot w(S) \leq w(\mathcal{C}^*)$, where $\mathcal{C}^*$ is an optimal set cover.

- How can we turn this fractional solution into an integer one?
  - i.e., we need to round the fractional values $x_S \in [0,1]$ to $\hat{x}_S \in \{0,1\}$
- First consider the setting with bounded element frequency f

Fractional Set Cover

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Set Cover: Randomized Rounding

Set Cover Rounding Algorithm:

1. Set $p_S \coloneqq \min\{1, x_S \cdot \ln \Delta\}$
2. Add each set S to set cover $\mathcal{C}$ with probability p_S (independently)
3. For each $e \in E$: If e is not covered, add min-weight set cont. e

Theorem: Given an optimal fractional weighted set cover solution, the set cover rounding algorithm computes a set cover $\mathcal{C}$ of expected weight

$$\mathbb{E}[\boldsymbol{w}(\boldsymbol{\mathcal{C}})] \leq \boldsymbol{w}(\boldsymbol{\mathcal{C}}^*) \cdot (\mathbf{1} + \mathbf{ln}\,\boldsymbol{\Delta})$$

Proof:

Set Cover: Randomized Rounding

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$$\mathbb{E}[\boldsymbol{w}(\boldsymbol{\mathcal{C}})] \leq \boldsymbol{w}(\boldsymbol{\mathcal{C}}^*) \cdot (\mathbf{1} + \ln \boldsymbol{\Delta})$$

Proof: We already know that

$$\mathbb{E}[X] \leq w(\mathcal{C}^*) \cdot \ln \Delta \quad \text{and} \quad \forall e \in E : q_e \leq \frac{1}{\Delta}$$

Set Cover Dual LP

Linear Program

$$\min \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t. } A\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual Linear Program

$$\max \boldsymbol{b}^T \boldsymbol{y}$$
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Set Cover: Randomized Rounding

Theorem: Given an optimal fractional weighted set cover solution, the set cover rounding algorithm computes a set cover $\mathcal{C}$ of expected weight

$$\mathbb{E}[w(\mathcal{C})] \leq w(\mathcal{C}^*) \cdot (1 + \ln \Delta)$$

Proof:

- It remains to show that

$$\sum_{e \in E} \frac{1}{\Delta} \cdot \min_{S : e \in S} w(S) \leq w(\mathcal{C}^*)$$

Approximating Weighted Vertex Cover

Recall maximal matching approximation for the unweighted case

- Vertex cover $S =$ all matched vertices of a maximal matching M
- S is a vertex cover because of the maximality of M
- Edges in M need to be covered by different nodes in $S^* \Longrightarrow |M| \leq |S^*|$

Generalization to Weighted Vertex Cover?

- The same algorithm does obviously not work
- Different view of above algorithm:

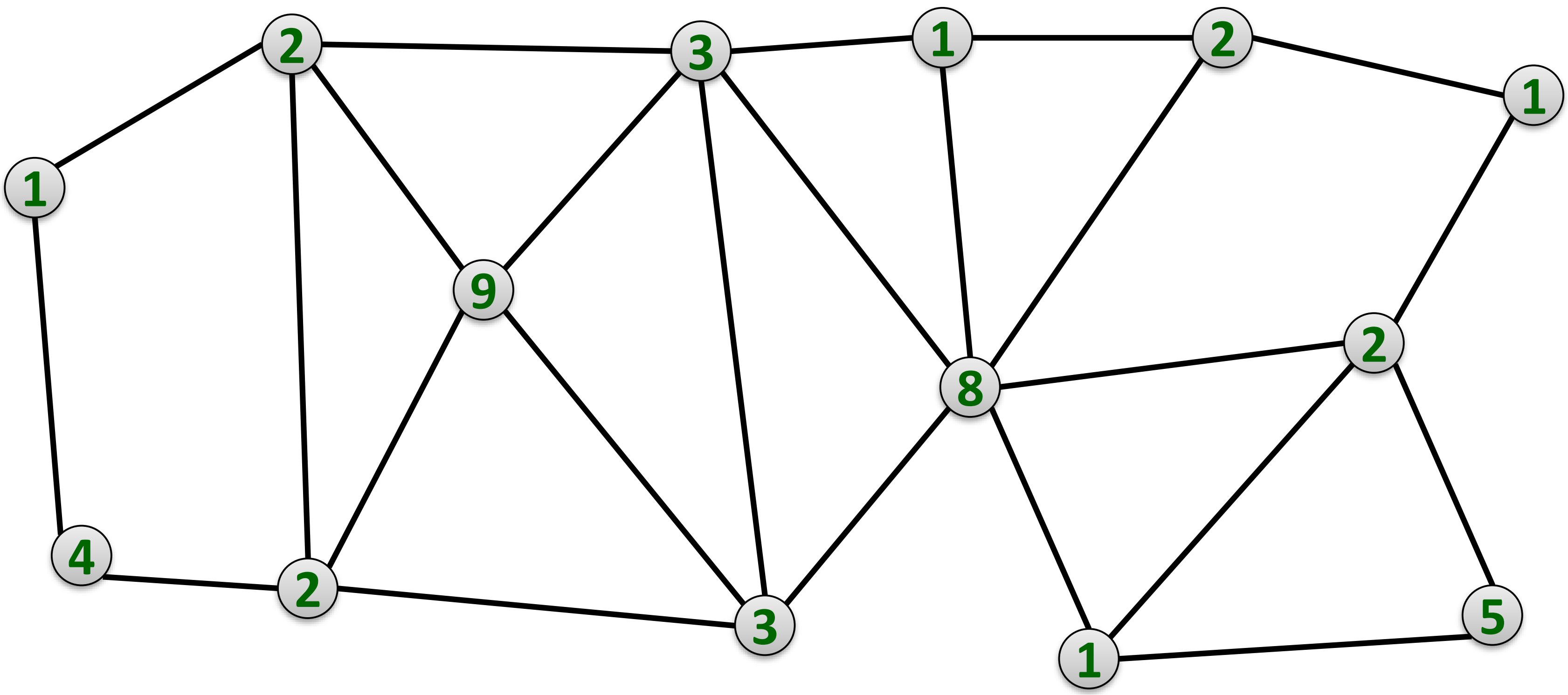
  Maximal matching M is a maximal feasible solution of the dual LP

Approximating Weighted Vertex Cover

Theorem: Let $y = \{y_e \geq 0 : e \in E\}$ be a maximal feasible solution of the dual weighted (hypergraph) vertex cover LP. Define the vertex set S as $S \coloneqq \{v \in V : \sum_{e:v\in e} y_e = w(v)\}$. Then, S is a vertex cover of weight

$$w(S) \leq f \cdot w(S^*).$$

Let's start with an example with $f = 2$:



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$$w(S) \leq f \cdot w(S^*).$$

Proof: