Tree Embeddings

Freitag, 10. Mai 2019 12:44

Approximating Graphs / Metric Spaces by Trees

Goal: Given a graph $G=(V,E,w)$, $(w(e) \geq 0)$
approximate the shortest path distances in G by a tree T.

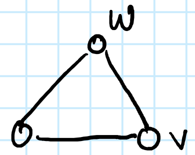
why? - simpler representation of G
- many problems are easier on trees

examples? if time, at the end of class
otherwise: exercises & next week's lecture

How can we approximate graph distances by a tree?

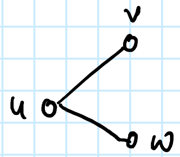
Examples

$K_3 = C_3$



$d(u,v) = d(u,w) = d(v,w) = 1$

spanning tree T:



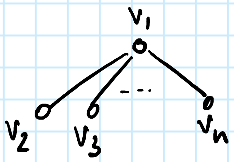
$d_T(u,v) = d_T(u,w) = 1$
$d_T(v,w) = 2$

T is dominating G

stretch: (assumption: $\forall u,v \in V: \; d_T(u,v) \geq d(u,v)$)

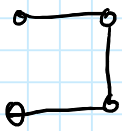
$\hookrightarrow \max\limits_{u,v \in V} \dfrac{d_T(u,v)}{d(u,v)}$ here: stretch $= 2$

K_n? repr. by a tree?



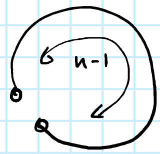
stretch $= 2$

C_4:



stretch $= 3$

C_n:

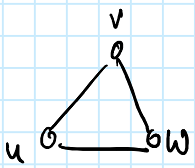


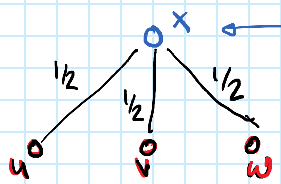
stretch $= n-1$

Can we do better?

We do not need to use a spanning tree

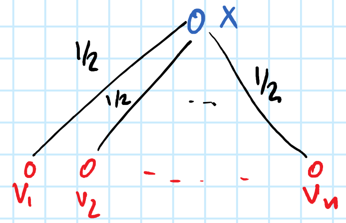
C_3:





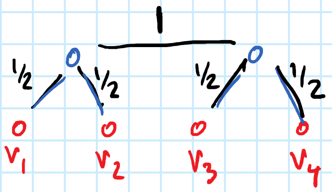
Steiner nodes

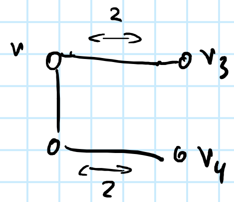
K_n:



K_n : best Steiner tree embedding has stretch 1

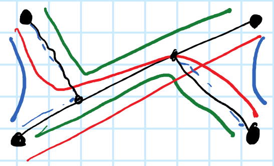
C_4:



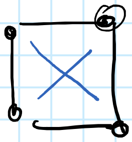


stretch $= 2$

stretch 2 is best possible for C_4



4 points condition



blue sum ≥ 4

C_n: not obvious, but for C_n, $\Omega(n)$ stretch is best possible
(follows from a more general result by [Rabinovich, Raz; 1998])

Probabilistic Tree Embedding

Goal: Find a collection of dominating trees $\mathcal{T} = \{T_1, \dots, T_N\}$ and a probability distribution $p_1, \dots, p_N$ $(\sum p_i = 1)$ such that

For a random tree $T \in \mathcal{T}$: $\forall u,v \in V$: $\mathbb{E}[d_T(u,v)] \leq \alpha \cdot d(u,v)$
according to distr. given by p_i

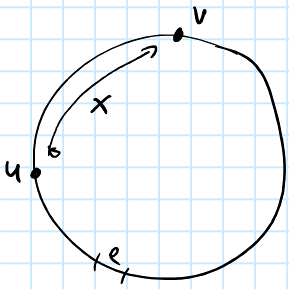
α: stretch of prob. embedding

Examples C_n

$T_1, \dots, T_n$ uniform distr. on spanning trees $T_1, \dots, T_n$ of C_n

$$\mathbb{E}\left[\frac{d_T(u,v)}{d(u,v)}\right] = \frac{n-x}{n} \cdot 1 + \frac{x}{n} \cdot \frac{n-x}{x}$$

$$= 2 \cdot \frac{n-x}{n} \leq 2$$



In the following, assume that we are given a metric space (V,d), $|V|=n$

(assume, w.l.o.g., $\min_{u,v} d(u,v) = 1$)

goal: find a prob. (Steiner) tree embedding of (V,d)

Def: (diameter)

For any $S \subseteq V$: $\mathrm{diam}(S) := \max_{u,v\in S} d(u,v)$

General Idea: ball carving

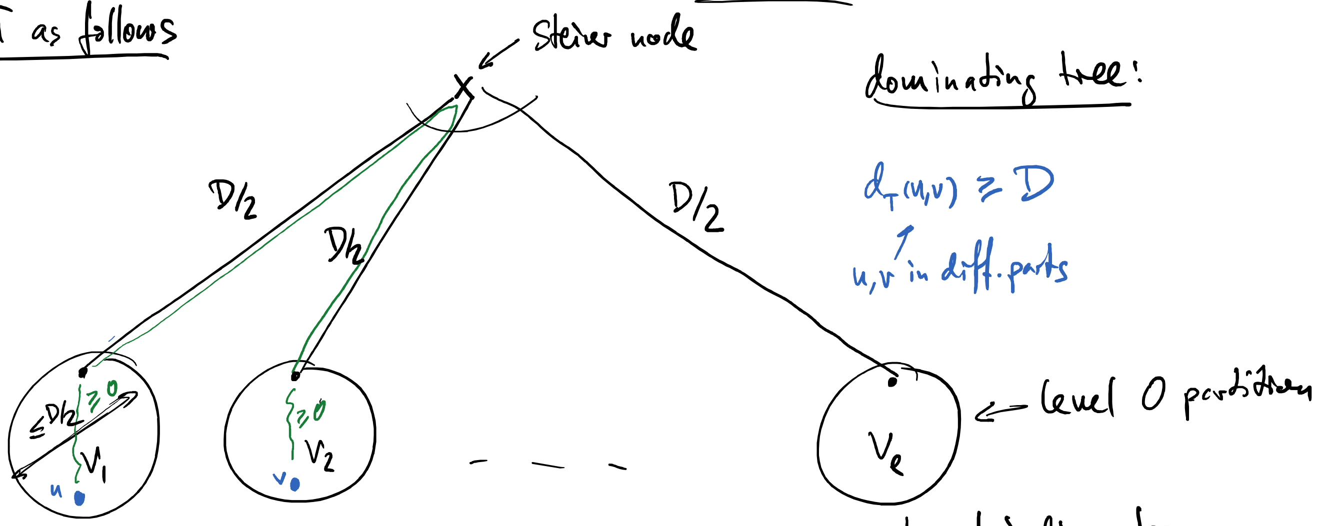
basic building block:

Given (V,d) with $\mathrm{diam}(V) \leq D$, partition V into $V_1,\dots,V_\ell$ s.t.

(a) $\forall i \in \{1,\dots,\ell\}$: $\mathrm{diam}(V_i) \leq D/2$

(b) $\forall u,v \in V$: $\mathbb{P}(u \,\&\, v \text{ are in diff. sets } V_i \,\&\, V_j) \leq \alpha \cdot \frac{d(u,v)}{D}$

Build tree T as follows



dominating tree:

$d_T(u,v) \geq D$

u,v in diff. parts

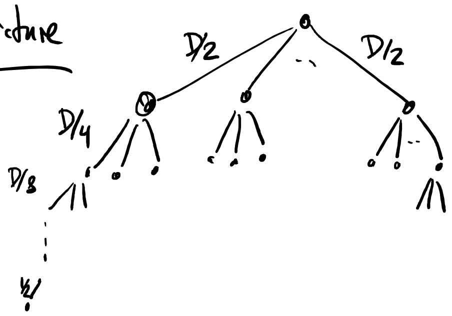
Lemma: If T separates u and v on level i,

then $d_T(u,v) = \left(\frac{D}{2^{i+1}} + \frac{D}{2^{i+2}} + \dots + \frac{D}{2^{\log_2 D}}\right) \leq D/2^i = D_i$

Level i diameter

$D_i := D/2^i$

Proof by picture



hierarchically well separated

$d_T(u,w) \leq \max\{d_T(u,v), d_T(v,w)\}$

Expected distance between u and v

$\mathcal{E}_i$: u & v are sep. on level i

$$\mathbb{E}[d_T(u,v)] \leq \sum_{i=0}^{\log D} \mathbb{P}(\mathcal{E}_i) \cdot D_i$$

$$\left(= \sum_{i=0}^{\log D} \mathbb{P}(\mathcal{E}_i) \cdot D/2^i \right)$$

$$\leq \sum_{i=0}^{\log D} \left(\alpha \cdot \frac{d(u,v)}{D_i} \right) \cdot D_i = \alpha \cdot (\log D + 1) \cdot d(u,v)$$

$\alpha = O(\log n)$

stretch: $O(\log n) \cdot O(\log D)$

Getting a partitioning with $\alpha = O(\log n)$

Goal: Given (V,d) with $\mathrm{diam}(V) \leq D$

Find partition $V_1, \dots, V_\ell$ of V s.t. (a) $\forall i : \mathrm{diam}(V_i) \leq D/2$

(b) $\forall u,v : \mathbb{P}(u,v \text{ sep. by part.}) \leq \alpha \cdot \frac{d(u,v)}{D}$

Ball Carving

$B(x,r) := \{y \in V : d(x,y) \leq r\}$

If $|V| = 1$ → return V

else:

$\beta \in_R [1,2]$

pick random permutation π on V → ($\pi(v)$ → position of v in ordering)

v_i : i^{th} node in random order

for $i = 1$ to n do

$V_i := B(v_i, \beta \cdot D/8) \setminus \bigcup_{j=1}^{i-1} V_j$

return the non-empty sets V_i

Lemma: $\forall i \in \{1, \dots, \ell\}$: $\mathrm{diam}(V_i) \leq D/2$ ✓

Fix two nodes u & v

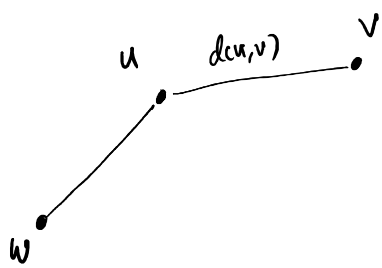
10 May 2019 09:40

Need to show that $\mathbb{P}(u \,\&\, v \text{ are sep.}) \leq \alpha \cdot \frac{d(u,v)}{D}$

Random variable $X^w_{u,v} = \begin{cases} D & \text{if } \{u,v\} \text{ are sep. by node } w \\ 0 \end{cases}$

$X_{u,v} = \sum X^w_{u,v} \quad \leftarrow \quad X_{u,v} = D \iff u \,\&\, v$ are sep.

assume $d(w,u) \leq d(w,v)$



$Y \leftarrow$ uniform rand. variable in $[D/8, D/4]$

$X^w_{u,v} = D$

$\hookrightarrow d(w,u) \leq \overbrace{\beta \cdot \frac{D}{8}}^{Y} < d(w,v) \leq d(w,u) + d(u,v)$

for all w' before w in random ordering:

$\min\{d(w',u), d(w',v)\} > \beta \cdot \frac{D}{8}$

Let's order all nodes according to distance to $\{u,v\}$

$w_1, w_2, \ldots, w_s$ $\quad X^{w_s}_{u,v} = D \quad$ $Y \in [D/8, D/4]$ $\quad \frac{D}{8} \quad d(u,v) \quad \frac{D}{4}$

$\hookrightarrow d(w_s,u) \leq \beta \cdot \frac{D}{8} < d(w_s,u) + d(u,v)$

$\hookrightarrow w_s$ appears before $w_1, \ldots, w_{s-1}$ in perm. π

$$\mathbb{P}(X^{w_s}_{u,v} = D) \leq \frac{1}{s} \cdot \frac{d(u,v)}{D/8} = \frac{8}{s} \cdot d(u,v) \cdot \frac{1}{D}$$

$$\mathbb{P}(X_{u,v} = D) = \sum_w \mathbb{P}(X^w_{u,v} = D) = \sum_{s=1}^{n} \mathbb{P}(X^{w_s}_{u,v} = D) \leq 8 \cdot \frac{d(u,v)}{D} \cdot \underbrace{\sum_{s=1}^{n} \frac{1}{s}}_{\leq \ln n + 1}$$

$$= \frac{O(\log n)}{D} \cdot d(u,v)$$

$$X^{w,i} = \begin{cases} D_i = D/2^i & \text{if } u \,\&\, v \text{ are sep. by } w \text{ on level } i \\ 0 & \text{otherwise} \end{cases}$$

We have seen that if u & v are cut on level i, $d_T(u,v) \leq D_i$

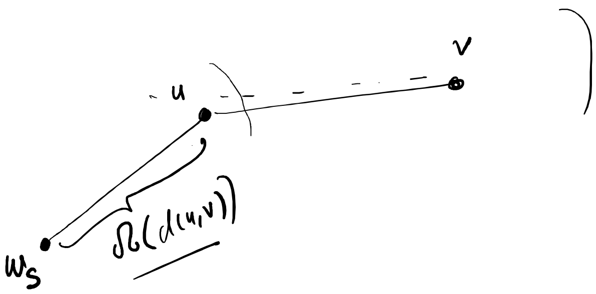
$$d_T(u,v) \leq \underbrace{\sum_{i=0}^{\log D} \sum_{w \in V} X_{u,v}^{w,i}} = \sum_{s=1}^{n} \boxed{\sum_{i=0}^{\log D} X_{u,v}^{w_s,i}}$$

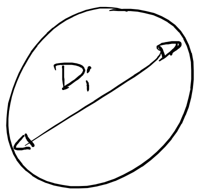
$$\mathbb{E}\left[X_{u,v}^{w_s,i}\right] \leq \frac{8}{s} \cdot d(u,v)$$

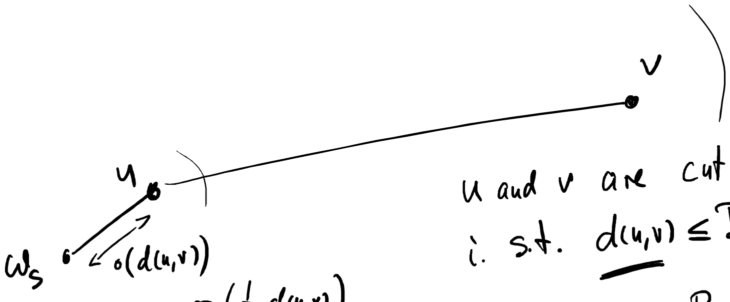
$$\mathbb{E}\left[d_T(u,v)\right] = \sum_{i=0}^{\log D} \sum_{s=1}^{n} \mathbb{E}\left[X_{u,w}^{w_s,i}\right] \leq O(\log D \cdot \log n) \cdot d(u,v)$$

$$R_i := \beta \cdot \frac{D_i}{8} \qquad \left(D_i = \frac{D}{2^i}\right)$$

$$R_{i+1} = 2 \cdot R_i$$







u and v are cut / split on a level i s.t. $d(u,v) \leq D_i$

$$R_i \geq \frac{D_i}{8} = \Omega(d(u,v))$$

$$\mathbb{E}\left[d_T(u,v)\right] = \sum_{s=1}^{n} \overbrace{\sum_{i=0}^{\log D} \underbrace{\mathbb{E}\left[X_{u,v}^{w_s,i}\right]}_{\leq \frac{8}{s} \cdot d(u,v)}}^{O\left(\frac{1}{s} \cdot d(u,v)\right)} = O(\log n) \cdot d(u,v)$$

$=0$ for all but $O(1)$ many i's