Graph Spanners

Freitag, 28. Juni 2019 09:40

Today & next week: graph sparsification

Goal: Given graph $G=(V,E)$, represent G by a sparser graph while preserving some properties

Today: preserving distances

Definition ((α,β)-spanner): A (α,β)-spanner of $G=(V,E)$ is a graph $G'=(V,E')$ with $E'\subseteq E$ s.t.

$$\forall u,v\in V:\ d_G(u,v) \le d_{G'}(u,v) \le \underbrace{\alpha}_{\text{multiplicative stretch}}\cdot d_G(u,v) + \underbrace{\beta}_{\text{additive stretch}}$$

α-multiplicative spanners

Girth of G: length of shortest cycle $(g(G))$

Obs: $g(G)>2k$: every k-hop neighborhood in G is a tree

Lemma: Let $G=(V,E)$ be an n-node graph with girth $g\ge 2k+1$.
Then $|E|\le 2\cdot n^{1+1/k}$. $(|E|\le 2\cdot n\cdot\lceil n^{1/k}\rceil)$

Proof: For contradiction, assume $|E|>2\cdot n\cdot\lceil n^{1/k}\rceil$

I) Transform $G\to G'$ with minimum degree $\ge\lceil n^{1/k}\rceil+1$

as long as there is a node of degree $\le\lceil n^{1/k}\rceil$ remove such a node

II) Assume $G'=(V',E')$

Consider some $v\in V'$ and consider the k-hop neighborhood of v

$$|V| = n \ge |V'| \ge \text{"\#nodes in } k\text{-hop neighborhood of } v\text{"}$$

$$> 1 + \sum_{i=1}^{k}(n^{1/k}+1)(n^{1/k})^{i-1}$$

$$= 1 + (n^{1/k}+1)\cdot\sum_{j=0}^{k-1}(n^{1/k})^j \qquad \left[\sum_{j=0}^{k-1} q^j = \frac{q^k-1}{q-1}\right]$$

$$= 1 + (n^{1/k}+1)\cdot\frac{(n^{1/k})^k-1}{n^{1/k}-1} = 1 + \underbrace{\frac{n^{1/k}+1}{n^{1/k}-1}}_{>1}(n-1)$$

$$> n$$

Multiplicative Spanner Construction

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Theorem: For every integer $k \geq 1$, every graph G has a $(2k-1)$-multiplicative spanner with $O(n^{1+1/k})$ edges.

Proof: Greedy construction

Initialize $E' = \emptyset$

Go through E in some order

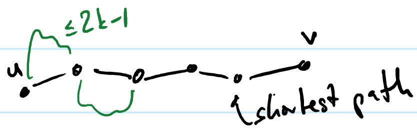
when considering edge $e = \{u, v\}$:

If $d_{G'}(u,v) \geq 2k$ then $E_s = E_s \cup \{e\}$

Sketch of G':

For every edge $\{u, v\} \in E$: $d_{G'}(u,v) \leq 2k-1$ ✓

For other pairs $u, v \in V$:

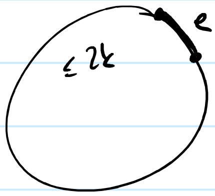


#edges of G':

construction guarantees that $g(G') \geq 2k+1$

assume otherwise

G' contains cycle of length $\leq 2k$



Conjecture [Erdős '64] For every fixed $k \geq 1$, there exists a family of graphs on n nodes with girth at least $2k+1$ and $\Omega(n^{1+1/k})$ edges.

Additive Spanners

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Let's look at some basic properties of vertex sampling

Graph $G=(V,E)$, choose set $S \subseteq V$ by including every $v \in V$ independently with prob. p.

(I) a) $\mathbb{E}[|S|] = n \cdot p$

b) $\mathbb{P}(|S| \geq 2 \cdot \mathbb{E}[|S|]) \leq e^{-\mathbb{E}[|S|]/3}$ (by Chernoff)

(II) $\forall v \in V$ and $N(v) := \{u \in V : \{u,v\} \in E\}$

a) $\mathbb{E}[|S \cap N(v)|] = p \cdot \deg(v)$

b) $\mathbb{P}(|S \cap N(v)| \geq 2 \cdot p \cdot \deg(v)) \leq e^{-p \cdot \deg(v)/3}$

c) $\mathbb{P}(|S \cap N(v)| = 0) = (1-p)^{\deg(v)} < e^{-p \cdot \deg(v)}$

Theorem: Every graph G has a 2-additive spanner with $\tilde{O}(n^{3/2})$ edges.

Remark $\tilde{O}(\cdot)$ hides polylogarithmic factors
$$\tilde{O}(f(n)) = f(n) \cdot (\log f(n))^{O(1)}$$

Proof:

Construction: Partition node set V in light nodes V_L and heavy nodes V_H

$$V_L := \{v \in V : \deg(v) \le \sqrt{n}\}, \quad V_H := V \setminus V_L$$

1. E_1' : set of all edges incident to some node in V_L
2. Initialize $E_2' = \emptyset$
   - Choose $S \subseteq V$ by indep. sampling each node with prob. $\frac{4 \ln n}{\sqrt{n}}$
   - For each $s \in S$, add a BFS tree rooted at s to E_2'

Spanner edges $E' = E_1' \cup E_2'$

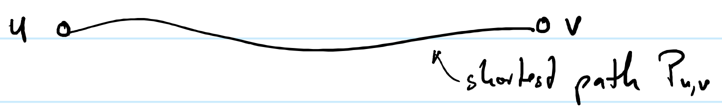
Number of edges

1. $|E_1'| \le n \cdot \sqrt{n} = n^{3/2}$
2. $|E_2'| \le n \cdot |S|$

$$\mathbb{E}(|S|) = n \cdot \frac{4 \ln n}{\sqrt{n}} = 4 \cdot \sqrt{n} \cdot \ln n \Rightarrow \mathbb{E}[E_2'] \le 4 n^{3/2} \cdot \ln n$$
$$\mathbb{P}(|S| \ge 8\sqrt{n} \ln n) < e^{-4\sqrt{n}\ln n/3} < n^{-\sqrt{n}}$$

$\Rightarrow$ With high prob. $|E'| = O(n^{3/2} \log n)$

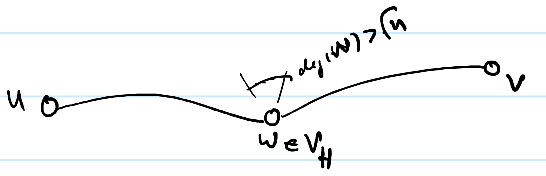
Additive stretch



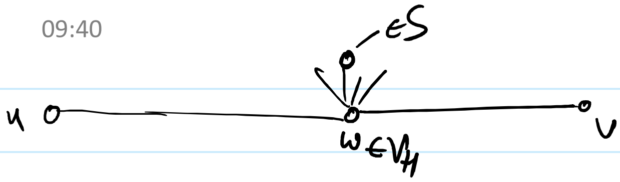
2 cases: (i) #heavy nodes in $P_{u,v} \le 1$

$\Rightarrow$ then $P_{u,v}$ is part of E_1'

(ii) #heavy nodes in $P_{u,v} \ge 1$



Recall:
S contains each node with prob. $\frac{4 \ln n}{\sqrt{n}}$

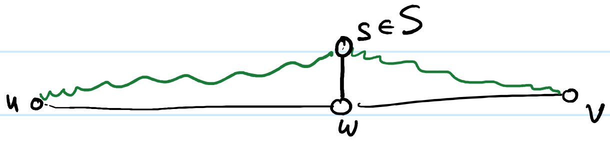


$$\left(e^{-\ln n}\right)^4 = \frac{1}{n^4}$$

$\deg(w) > \sqrt{n}$

$$\mathbb{P}(|S \cap N(w)| = 0) < e^{-\frac{4\cdot\ln n}{\sqrt{n}}\sqrt{n}} = \frac{1}{n^4}$$

$\Rightarrow$ every heavy node has a sampled neighbor with prob. $\geq 1 - \frac{1}{n^3}$



$$\begin{aligned} d_{G'}(u,v) &\leq d_{G'}(u,s) + d_{G'}(v,s) \\ &= d_G(u,s) + d_G(v,s) \\ &\leq d_G(u,w) + \underbrace{d_G(w,s)}_{=1} + \underbrace{d_G(w,s)}_{=1} + d_G(w,v) \\ &= d_G(u,v) + 2 \end{aligned}$$

<u>Theorem</u> Every graph G has a 4-additive spanner with $\hat{O}(n^{7/5})$ edges.

<u>Proof:</u> $V_L := \{v \in V : \deg(v) \leq n^{2/5}\}$, $V_H := V \setminus V_L$

1.) E_1' : set of edges incident ~~to~~ some $v \in V_L$

2.) Initialize $E_2' = \emptyset$
- choose S by sampling each $v \in V$ with prob. $30 \cdot \frac{\ln n}{n^{3/5}}$
- add BFS tree for each $s \in S$ to E_2'

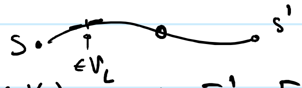
3.) Initialize $E_3' = \emptyset$
- choose S' by indep. sampling each $v \in V$ with prob. $10 \cdot \frac{\ln n}{n^{2/5}}$,
- For each heavy node $w \in V_H$, add edges $\{w, s'\}$ for nodes $s' \in S'$
- For each $s, s' \in S'$, add a shortest path $\underline{P_{s,s'}}$ between s and s' with at most $\underline{n^{1/5}}$ heavy nodes to E_3'

<u>Size of $E_3' \setminus E_1'$:</u>

~~for~~ $|S'| = \Theta(n^{3/5} \cdot \log n)$

#pairs $s, s' \in S'$ $\to O(n^{6/5} \log^2 n)$

for each pair



↳ add $O(n^{1/5})$ edges in $E_3' \setminus E_1'$

$|E_3'| = O(n^{7/5} \log^2 n)$

Additive Stretch ≤ 4

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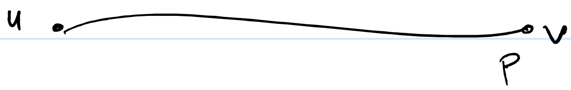


(i) all edges of P are incident to a light node ✓

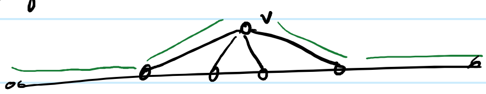
(ii) # heavy nodes on P $> n^{1/5}$

at least one heavy node $w \in P$ has a neighbor in S

each heavy node has $> n^{2/5}$ neighbors

→ in total, the heavy nodes on P have $> n^{3/5}$ neighbors (ignoring double counting)

→ no node can have more than 3 neighbors on P



(iii) # heavy nodes on P is between 2 and $n^{1/5}$