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Advanced Algorithms
Sample Solution Problem Set 1

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Exercise 1: Set Cover Integrality Gap

Consider some combinatorial optimization problem $\mathcal{P}$ that can be phrased as an integer linear program (ILP) and let $\mathcal{P}_f$ be the corresponding LP relaxation. The *integrality gap* of such a problem $\mathcal{P}$ is defined as the maximum possible ratio between the value of an optimal solution for $\mathcal{P}$ and the value of an optimal solution for the LP relaxation $\mathcal{P}_f$. The integrality gap is the best possible approximation guarantee one can hope for when using LP-based methods to design/analyze an approximation algorithm. The goal of this exercise is to bound the integrality gap of the set cover problem for the LP relaxation that we considered in the lecture.

(a) Show that for every integer $f \geq 2$, there exists an unweighted set cover instance $(E, \mathcal{S})$ with maximum element frequency f for which the integrality gap is equal to at least $f - \varepsilon$ for any (arbitrarily small) constant $\varepsilon > 0$.

(b) Show that there exists a constant $c > 0$ such that for every sufficiently large positive integer m, there exists an unweighted set cover instance $(E, \mathcal{S})$ with $|E| = m$ elements for which the integrality gap is at least $c \ln m$.

Sample Solution

(a) We have seen in the lecture that finding a minimum set cover of $(E, \mathcal{S})$ is equivalent to finding a vertex cover in the hyper-graph $G = (V, E')$ $(V := \mathcal{S}$ and $E' := \{e_x \mid x \in E\}$ with $e_x := \{S \in \mathcal{S} : x \in \mathcal{S}\})$. That means, proving an integrality gap for the vertex cover problem on hyper graphs shows the same thing for the set cover problem. Our idea is to use a complete uniform[1] hyper-graph $G = (V, E')$ of rank f with $|V| = n$ chosen large enough.

We claim that any (integral) vertex cover $S^{\text{int}} \subseteq V$ of G has size at least $|S^{\text{int}}| \geq n - f + 1$. For a contradiction, assume that there would be f nodes $V \ni v_1, \ldots, v_f \notin S^{\text{int}}$. But then the hyper-edge $e = \{v_1, \ldots, v_f\}$ would not be covered.

For the fractional vertex cover problem we need to assign fractional values $x_v \in [0, 1]$ to each node $v \in V$, such that for every hyperedge $e \in E'$ we have $\sum_{v \in e} x_v \geq 1$. Since $|e| = f$ we achieve this simply by assigning the value of $x_v = \frac{1}{f}$ to each node v. The size of this fractional vertex cover $S^{\text{frc}} := \{x_v \mid v \in V\}$ of nodes is $|S^{\text{frc}}| := \sum_{v \in V} x_v = n/f$.

With a lower bound $|S^{\text{int}}| \geq n - f + 1$ for *any* integral vertex cover and an upper bound $|S^{\text{frc}}| \leq n/f$ for the *minimum* fractional vertex cover of the hypergraph G we can give the following lower bound for the integrality gap

$$\frac{n - f + 1}{n/f} = f - \frac{f(f-1)}{n} \geq f - \varepsilon, \quad \text{by choosing } n \geq \frac{f(f-1)}{\varepsilon}.$$

[1] A hypergraph $G = (V, E')$ is uniform of rank f if every edge $e \in E'$ has cardinality exactly $|e| = f$.

(b) First of all, notice that m corresponds to the number of edges $|E'|$ in the dual hypergraph $G = (V, E')$. If we construct G like in part (a), we have

$$\sum_{i=1}^{n} \binom{n}{i} = 2^n \quad \Longrightarrow \quad m = \binom{n}{f} \leq 2^n \quad \Longrightarrow \quad n \geq \log_2 m. \tag{1}$$

(The first equality is due to the fact that the cardinality of the power set of a set of size n is 2^n, which equals number of all subsets of size k of a set of size n, summed over k).

We choose G such that $f = \lceil n/2 \rceil$ (i.e. we fix the rank of G to $\lceil n/2 \rceil$, c.f. construction given in part (a)). Let $\varepsilon = f/2$. Then $\frac{f(f-1)}{\varepsilon} \leq 2(f-1) \leq n$ holds and from part (a) we know that the integrality gap is at least

$$f - \varepsilon = \frac{f}{2} \geq \frac{n}{4} \overset{\text{Eq (1)}}{\geq} \frac{\log_2 m}{4} = \frac{\ln m}{4 \ln 2}.$$

Exercise 2: Minimum Membership Set Cover

We now consider the minimum membership set cover problem, which is a variation of the set cover problem. As in the minimum set cover problem, we want to compute a set cover $\mathcal{C}$ of a given set system $(E, \mathcal{S})$ with $n = |E|$ elements. However, instead of minimizing the cardinaliy $|\mathcal{C}|$ of $\mathcal{C}$, we want to minimize the maximum number of times an element is covered by a set in $\mathcal{C}$. Formally, we want to find a set cover $\mathcal{C}$ that minimizes

$$\max_{e \in E} \left| \{S \in \mathcal{C} \,:\, e \in S\} \right|.$$

(a) Show that it is NP-hard to approximate the minimum membership set cover problem within a factor $(1-\varepsilon)\ln n$ for any constant $\varepsilon > 0$.

Hint: You can use that it is NP-hard to approximate the (unweighted) minimum set cover problem within a factor $(1-\varepsilon)\ln n$ *for any constant* $\varepsilon > 0$.

(b) Phrase the above problem as an integer linear program.

(c) Show that by solving an appropriate LP relaxation of your ILP, combined with randomized rounding, one can obtain $O(\log n)$-approximation of the minimum membership set cover problem with probability at least $1/2$.

Sample Solution

(a) We do as we are told in the hint, and reduce the minimum set cover (approximation) problem (MSC) to the minimum membership set cover (approximation) problem (MMSC). Given an instance $(E, \mathcal{S})$ of MSC, we transform it into an instance $(E', \mathcal{S}')$ of MMSC. Let $E' = E \cup \{e\}$ where e is a new element that is not part of the original set E. Let $\mathcal{S}' := \{S \cup \{e\} \mid S \in \mathcal{S}\}$. This transformation can obviously be done in polynomial time with respect to the input size $|E| + |\mathcal{S}|$.

Let $\mathcal{C}$ be a (not necessarily optimal) set cover of $(E, \mathcal{S})$. Then $\mathcal{C}' := \{S \cup \{e\} \mid S \in \mathcal{C}\}$ is obviously a set cover of $(E', \mathcal{S}')$. Furthermore, $\mathcal{C}'$ has the maximum membership of $|\mathcal{C}'| = |\mathcal{C}|$ since e is covered by all sets in $\mathcal{C}'$. The same trick works in the other direction, i.e. given a solution $\mathcal{C}'$ of MMSC on $(E', \mathcal{S}')$ we can define $\mathcal{C} := \{S \setminus \{e\} \mid S \in \mathcal{C}'\}$ to obtain a solution where the maximum membership of $\mathcal{C}'$ (defined by the membership of e) equals the size $|\mathcal{C}'| = |\mathcal{C}|$.

That means that an algorithm to compute an $(1-\varepsilon)\ln n$ approximation for MMSC can be used to compute the same thing for MSC. But since $(1-\varepsilon)\ln n$ approximation of MSC is known to be NP-hard, so must be the computation of an $(1-\varepsilon)\ln n$ approximation of MMSC.

(b) We define variable $x_S \in \{0, 1\}$ for each $S \in \mathcal{S}$, where $x_S = \mathbb{1}_{\mathcal{C}}(S)$ represent the indicator function of $\mathcal{C}$. We introduce one additional variable $x \in \{0, 1\}$. We get the following ILP:

$$\min x \text{ s.t. } \forall e \in E : 1 \leq \sum_{S : e \in S} x_S \leq x.$$

(c) We adapt the algorithm and the analysis for randomized rounding from the lecture. First we solve the LP relaxation of the above ILP, which gives as an optimal fractional solution $\mathcal{C}^*$ that is at least as good as any integer solution $\mathcal{C}$. Then we set $p_S := \min(1, x_S \cdot c \cdot \ln n)$ and add each $S \in \mathcal{S}$ to $\mathcal{C}$ independently with probability p_S. We show that this gives us as an $O(\log n)$ approximation of $\mathcal{C}^*$ with high probability (w.h.p.).[2]

Let q_e be the probability that $e \in E$ is not covered. If $p_S = 1$ for some $S \in \mathcal{S}$ then obviously $q_e = 0$. It remains to consider the case $p_S = x_S \cdot c \cdot \ln n$ for all S with $e \in S$. Then

$$\begin{aligned} q_e &\leq \prod_{S:e\in S} (1-p_S) \overset{(1-x\leq e^{-x})}{\leq} \prod_{S:e\in S} e^{-p_S} = \exp\Big(-\sum_{S:e\in S} p_S\Big) \\ &= \exp\Big(-c\ln n \sum_{S:e\in S} x_S\Big) \overset{(\sum_{S:e\in S} x_S \geq 1)}{\leq} e^{c \ln n} = \frac{1}{n^c}. \end{aligned}$$

Let $X = |\mathcal{C}|$ be the random number of sets $S \in \mathcal{C}$. We have $\mathbb{E}(X) = \sum_{S\in\mathcal{C}} p_S \geq c \ln n \sum_{S\in\mathcal{C}} x_S$. We bound the probability that X is more than four times larger than the expected size with a Chernoff bound.[3]

$$\mathbb{P}\big(X \geq (1+3)\mathbb{E}(X)\big) \leq \exp\Big(-\frac{3\mathbb{E}(X)}{3}\Big) \leq \exp\big(-c \cdot \ln n \sum_{S\in\mathcal{C}} x_S\big) \leq \exp\big(-c \cdot \ln n\big) = \frac{1}{n^c}.$$

That means that each element $e \in E$ is covered in $\mathcal{C}$ by at most $4c \ln n \in O(\log n)$ times the number of sets as in the optimal fractional solution $\mathcal{C}^*$ w.h.p.

For an element $e \in E$ let $\mathcal{E}_e^1$ be the event that e is covered by $\mathcal{C}$ and $\mathcal{E}^2$ the event that $|\mathcal{C}|$ is at most $O(\log n)$ times bigger than $w(\mathcal{C}^*)$. We showed that both events occur w.h.p. Then the lemma below shows that $\big(\bigcap_{e\in E} \mathcal{E}_e^1\big) \cap \mathcal{E}^2$ occurs w.h.p. as well. We choose the hidden constant in the "w.h.p." sufficiently large such that the overall probability of success is at least 1/2.

Lemma 1. *Let $E_1, \ldots, E_k$ be events each taking place w.h.p. If $k \leq p(n)$ for a polynomial p then $E := \bigcap_{i=1}^k E_i$ also takes place w.h.p.*

Proof. Let $d := \deg(p)+1$. Then there is an $n_0 \geq 0$ such that $p(n) \leq n^d$ for all $n \geq n_0$. Let $c > 0$ and $n_1, \ldots, n_k \in \mathbb{N}$ such that for all $i \in \{1, \ldots, k\}$ we have $\mathbb{P}(\overline{E_i}) \leq \frac{1}{n^c}$. With Boole's Inequality (union bound) we obtain

$$\mathbb{P}(\overline{E}) = \mathbb{P}\Big(\bigcup_{i=1}^k \overline{E_i}\Big) \leq \sum_{i=1}^k \mathbb{P}(\overline{E_i}) \leq \sum_{i=1}^k \frac{1}{n^c} \leq \frac{1}{n^{c-d}}$$

for all $n \geq n_0' := \max(n_0, \ldots, n_k)$. Let $c' > 0$ be arbitrary. We choose $c := c' + d$. Then we have $\mathbb{P}\big(\overline{E}\big) \leq \frac{1}{n^{c'}}$ for all $n \geq n_0'$. $\square$

Exercise 3: Minimum Weighted Set Double Cover

Next, we consider a variation of the weighted set cover problem that we call the minimum weighted set double cover. Assume that we are given a set system $(E, \mathcal{S})$ such that each element $e \in E$ is contained in at least two different sets in $\mathcal{S}$. Further, each set $S \in \mathcal{S}$ is assigned a positive weight $w(S) > 0$. The goal now is to choose a collection $\mathcal{C} \subseteq \mathcal{S}$ of sets such that each element $e \in E$ is contained in at least 2 of the sets in $\mathcal{C}$. The total weight should be as small as possible.

(a) Phrase the above problem as an integer linear program.

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[2] That means that for arbitrary $c > 0$ the probability is at least $1 - \frac{1}{n^c}$.

[3] For $\delta \geq 1$ it holds that $\mathbb{P}\big(X \geq (1+\delta)\mathbb{E}(X)\big) \leq \exp\Big(-\frac{\delta\mathbb{E}(X)}{3}\Big)$.

(b) Let $n = |E|$ be the number of elements. Show that by solving an appropriate LP relaxation of your ILP, combined with randomized rounding, one can obtain $O(\log n)$-approximation of the minimum weighted set double cover problem with probability at least $1/2$.

Hint 1: You need to make sure that the variables of your LP only take values between 0 *and* 1*.*

Hint 2: Choose the probabilities for the rounding step sufficiently large so that the last step of the rounding algorithm from the lecture is not necessary.

(c*) Let Δ be the largest set size. Show that by slightly adapting the randomized rounding algorithm for the usual minimum weighted set cover problem from the lecture, you can obtain an approximation algorithm for the minimum weighted set double cover problem with expected approximation ratio at most $\ln(2\Delta) + 1$.

Hint: Note that this subproblem is significantly more challenging. Changing the LP such that the variables can only take values between 0 *and* 1 *also changes the structure of the dual LP, which makes the last step of the rounding algorithm of the lecture more tricky.*

Sample Solution

(a) Let $\mathbf{x} = (x_S)_{S\in\mathcal{S}}$ where the $\{0,1\} \ni x_S = \mathbb{1}_{\mathcal{C}}(S)$ represent the values of the indicator function of a solution $\mathcal{C}$ and $\mathbf{w} = (w(S))_{S\in\mathcal{S}}$. We set up the following ILP

$$\min \mathbf{w}^T\mathbf{x} \quad \text{s.t. } \forall S \in \mathcal{S} : 0 \le x_S \le 1 \text{ and } \forall e \in E : \sum_{S:e\in S} x_S \ge 2.$$

(b) As in exercise 2 (c) we solve the above ILP as LP and set $p_S := \min(1, x_S \cdot c \cdot \ln n)$. Then we add each S to $\mathcal{C}$ independently with probability p_S.

Let $e \in E$. We show that e is at least doubly covered. In contrast to exercise 2 (c), we show that the probability for the following two events is smaller than $1/n^c$ for any $c > 0$. Let $\mathcal{E}_e^0$ be the event that e is covered by no set at all. We already showed that $\mathbb{P}(\mathcal{E}_e^0) \le 1/n^c$ in exercise 2 (c), which works analogously here.

Let $\mathcal{E}_e^1$ be the event that e is covered by exactly one set. For some $S' \in \mathcal{S}$ with $e \in S'$ we first analyze the conditional probability $\mathbb{P}(\mathcal{E}_e^1 \mid S' \in \mathcal{C})$. Let $\mathcal{T} := \{S \in \mathcal{S} \mid e \in S, S \ne S'\}$ be the sets that contain e except S'.

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_e^1 \mid S' \in \mathcal{C}) &\le \prod_{S\in\mathcal{T}}(1-p_S) \le \exp\Big(-\sum_{S\in\mathcal{T}} p_S\Big) = \exp\Big(-c\ln n \sum_{S\in\mathcal{T}} x_S\Big) \\
&\le \exp\Big(-c\ln n\big([\sum_{S:e\in S} x_S] - x_{S'}\big)\Big) \overset{x_{S'}\le 1}{\le} \exp\Big(-c\ln n\big([\sum_{S:e\in S} x_S] - 1\big)\Big) \\
&\overset{(\sum_{S:e\in S} x_S \ge 2)}{\le} \exp\left(-c\ln n\right) = \frac{1}{n^c}.
\end{aligned}$$

Using the law of total probability we obtain the following for the probability of $\mathcal{E}_e^1$

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_e^1) &= \sum_{S':e\in S'} \mathbb{P}(\mathcal{E}_e^1 \mid S' \in \mathcal{C}) \cdot \mathbb{P}(S' \in \mathcal{C}) \le \frac{1}{n^c} \sum_{S':e\in S'} p_{S'} \\
&\overset{(*)}{\le} \frac{c\ln n}{n^c}\Big(1 + \sum_{\substack{S':e\in S' \\ S'\ne \hat{S}}} x_S\Big) \overset{(**)}{\le} \frac{c\ln n}{n^c} \cdot (2n+1) \overset{(***)}{\le} \frac{n^2}{n^c} = \frac{1}{n^{c-2}} \overset{c':=c-2}{=} \frac{1}{n^{c'}}
\end{aligned}$$

Inequality $(*)$ is because we have $p_{\hat{S}} = 1$ for at most one $\hat{S} \in \mathcal{S}$ with $e \in \hat{S}$ (otherwise $\mathbb{P}(\mathcal{E}_e^1) = 0$ and we are done) and otherwise $p_{S'} = x_{S'} \cdot c \cdot \ln n$. Inequality $(**)$ is because for an *optimal, unweighted, integer* solution $\mathcal{C}'$ we have $|\mathcal{C}'| \le 2n$ (two sets per element $e \in E$ suffice to double cover every element). Hence $\sum_{S':e\in S'} x_S \le |\mathcal{C}'| \le 2n$ always suffices to cover all elements in the fractional case. Inequality $(***)$ holds for sufficiently large n. In summary $\overline{\mathcal{E}}_e^1$ occurs w.h.p.

Finally, let $\mathcal{E}^2$ be the event that our solution $\mathcal{C}$ does not fulfill $w(\mathcal{C}) = O(\log n)w(\mathcal{C}^*)$ for the optimal fractional set cover $\mathcal{C}^*$. Let $X = \sum_{S\in\mathcal{C}} w(S)$ be the random weight of our integer solution $\mathcal{C}$. We have

$$\mathbb{E}(X) = \sum_{S\in\mathcal{S}} p_S \cdot w(S) \ge c\ln n \sum_{S\in\mathcal{S}} x_S\, w(S) = c\ln n \cdot w(\mathcal{C}^*).$$

We bound the probability that X is more than four times larger than the expected size with a Markov bound.

$$\mathbb{P}\big(X \ge 4\mathbb{E}(X)\big) \le \frac{1}{4}$$

That means that $\mathcal{C}$ uses at most $4c\ln n \in O(\log n)$ times the weight of an optimal fractional solution $\mathcal{C}^*$ w.h.p., hence $\overline{\mathcal{E}}^2$ occurs w.h.p. Moreover, we have seen that $\overline{\mathcal{E}}_e^0, \overline{\mathcal{E}}_e^1$ occur w.h.p. for all $e \in E$. We invoke Lemma 1, which shows us that $\mathbb{P}\big(\bigcup_{e\in E} \mathcal{E}_e^0 \cup \mathcal{E}_e^1\big) \le \frac{1}{n^c}$. Finally, we adapt the hidden constant in the "w.h.p." such that

$$\mathbb{P}\big[\big(\bigcup_{e\in E} \mathcal{E}_e^0 \cup \mathcal{E}_e^1\big) \cup \mathcal{E}^2\big] \le \frac{1}{n^c} + \frac{1}{4} \le 1/2.$$

(c*) First we set up the LP in a matrix-vector form (to easily derive the dual LP later).

LP: Let $\mathbf{x} = (x_S)_{S\in\mathcal{S}}$ where the $\{0,1\} \ni x_S = \mathbb{1}_{\mathcal{C}}(S)$ and $\mathbf{w} = (w(S))_{S\in\mathcal{S}}$.

$$\min \mathbf{w}^T\mathbf{x} \quad \text{s.t.} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\mathbf{x} \le \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$$

where $A_1\mathbf{x} \le \mathbf{b}_1$ represents the inequalities $\forall e \in E : \sum_{S:e\in S} x_S \ge 2$ and $A_2\mathbf{x} \le \mathbf{b}_2$ represents $\forall S \in \mathcal{S} : x_S \le 1$. Note that $A_1 \in \mathbb{R}^{n\times m}$, $A_2 \in \mathbb{R}^{m\times m}$, $\mathbf{b_1} \in \mathbb{R}^n$, $\mathbf{b_2} \in \mathbb{R}^m$.

We solve the relaxed LP to obtain solutions $x_S \in [0,1]$. We round these to 1 randomly with probability $p_S := \min(1, x_S \cdot \ln(2\Delta))$. Let $\mathcal{C}$ be the resulting solution. That gives us

$$\mathbb{E}\big(w(\mathcal{C})\big) = \sum_{S\in\mathcal{S}} p_S w(S) \le \ln(2\Delta) \sum_{S\in\mathcal{S}} x_S w(S) = \ln(2\Delta) w(\mathcal{C}^*).$$

However we may have to add more sets in case an element is not yet covered at least twice. If $e \in E$ is not covered, we add the two sets S_1^e, S_2^e to $\mathcal{C}$ that are the first and second "cheapest" sets of all those sets containing e. If e is covered exactly once, we add S_2^e to $\mathcal{C}$. Now we must analyze the number of sets we add in this step. For that purpose, let us first analyze the probability that some $e \in E$ is covered by at most one set $S \in \mathcal{C}$.

$$\begin{aligned}
\mathbb{P}\big(|\{S\in\mathcal{C} \mid e\in S\}| \le 1\big) &= \mathbb{P}\big(|\{S\in\mathcal{C} \mid e\in S\}| = 0\big) + \mathbb{P}\big(|\{S\in\mathcal{C} \mid e\in S\}| = 1\big) \\
&= \prod_{S:e\in S}(1-p_S) + \sum_{S':e\in S'} p_{S'} \prod_{S:e\in S, S\neq S'}(1-p_S) \\
&\le \exp\Big(-\sum_{S:e\in S} p_S\Big) + \sum_{S':e\in S'} p_{S'} \cdot \exp\Big(-\sum_{S:e\in S, S\neq S'} p_S\Big) \\
&= \exp\Big(-\sum_{S:e\in S} p_S\Big) + \sum_{S':e\in S'} p_{S'} \cdot \exp\Big(p_{S'} - \sum_{S:e\in S} p_S\Big) \\
&= \exp\Big(-\sum_{S:e\in S} p_S\Big)\big(1 + \sum_{S':e\in S'} p_{S'} \cdot e^{p_{S'}}\big) \\
&\le \exp\Big(-\sum_{S:e\in S} p_S\Big)\big(1 + e\sum_{S':e\in S'} p_{S'}\big) \\
&= e^{-\xi_e}\big(1 + e\xi_e\big) \qquad\qquad \text{(where we define } \xi_e := \textstyle\sum_{S:e\in S} p_S)
\end{aligned}$$

The derivative of $e^{-\xi_e}\big(1+e\xi_e\big)$ is $e^{-\xi_e}\big(e-e\xi_e-1\big)<0$ which is smaller than 0 because $\xi_e \geq 1$. Therefore we maximize $e^{-\xi_e}\big(1+e\xi_e\big)$ by minimizing $\xi_e = \sum_{S:e\in S} p_S$. Since we have $\sum_{S:e\in S} x_S \geq 2$, the minimal value for ξ_e is $\sum_{S:e\in S} p_S = \ln(2\Delta)\sum_{S:e\in S} x_S \geq 2\ln(2\Delta)$. In conclusion we obtain

$$\mathbb{P}\big(|\{S\in\mathcal{C} \mid e\in S\}|\leq 1\big) \leq e^{-\xi_e}\big(1+e\xi_e\big) \leq \frac{1+2e\ln(2\Delta)}{4\Delta^2} \leq \frac{1}{2\Delta}$$

for large enough Δ ($\Delta \geq 9$ suffices).[4] That means in expectation we add at most

$$\sum_{e\in E}\mathbb{P}\big(|\{S\in\mathcal{C} \mid e\in S\}|\leq 1\big)(w(S_1^e)+w(S_2^e)) \leq \sum_{e\in E}\frac{w(S_1^e)+w(S_2^e)}{2\Delta}$$

much additional weight to $\mathcal{C}$. To analyze this further we derive the dual LP.

Dual LP: We define $\mathbf{y} := \binom{\mathbf{y_1}}{\mathbf{y_2}}$ and $\mathbf{y_1} := (y_e)_{e\in E}$ and $\mathbf{y_2} := (y_S)_{S\in\mathcal{S}}$.

$$\max \begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix}^T \begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix}A_1\\ A_2\end{pmatrix}^T \begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} \leq \mathbf{w}$$

where $\max \binom{\mathbf{b}_1}{\mathbf{b}_2}^T \binom{\mathbf{y}_1}{\mathbf{y}_2}$ is the same as $\max\big(2\sum_{e\in E} y_e - \sum_{S\in\mathcal{S}} y_S\big)$ and $\binom{A_1}{A_2}^T \mathbf{y} \leq \mathbf{w}$ is equivalent to $\forall S\in\mathcal{S}: \big(\sum_{e\in S} y_e\big) - y_S \leq w(S)$. Let us define

$$y_e := \frac{w(S_1^e)+w(S_2^e)}{2\Delta}, \quad y_S := \sum_{e:S=S_1^e} y_e.$$

Than this choice of y_e, y_S is a feasible solution for the dual LP since for all $S\in\mathcal{S}$:

$$\begin{aligned}
\big(\sum_{e\in S} y_e\big) - y_S &= \sum_{e\in S}\frac{w(S_1^e)+w(S_2^e)}{2\Delta} - \sum_{e:S=S_1^e}\frac{w(S_1^e)+w(S_2^e)}{2\Delta} \\
&\leq \sum_{e\in S}\frac{w(S_1^e)}{2\Delta} + \sum_{e\in S}\frac{w(S_2^e)}{2\Delta} - \sum_{e:S=S_1^e}\frac{w(S_2^e)}{2\Delta} \\
&\leq \sum_{e\in S}\frac{w(S)}{2\Delta} + \sum_{\substack{e\in S\\ S\neq S_1^e}}\frac{w(S_2^e)}{2\Delta} \\
&\leq 2\sum_{e\in S}\frac{w(S)}{2\Delta} = \frac{|S|w(S)}{\Delta} \leq w(S)
\end{aligned}$$

Then for our objective function in the dual LP we have due to weak duality

$$2\sum_{e\in E} y_e - \sum_{S\in\mathcal{S}} y_S = 2\sum_{e\in E} y_e - \sum_{S\in\mathcal{S}}\sum_{e:S=S_1^e} y_e = \sum_{e\in E} y_e \leq \sum_{S\in\mathcal{S}} x_S w(S) \leq w(\mathcal{C}^*).$$

[4] if we show only the looser bound $\mathbb{P}\big(|\{S\in\mathcal{C} \mid e\in S\}|\leq 1\big) \leq \frac{1}{c\Delta}$ for some $c<2$ we can make this work for $\Delta \geq 2$, which covers all interesting cases, but at the moderate cost of getting an $2/c + \ln\Delta$ approximation instead of $1+\ln\Delta$.