University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
P. Schneider

# Advanced Algorithms
# Sample Solution Problem Set 2

**Issued:** Friday May 3, 2019

## Exercise 1: Random Walk on a Line and on a $d$-dimensional Mesh

Consider a random walk on the integers. The walk starts at 0 and in each step, it either moves 1 to the right or 1 to the left, each with probability 1/2. That is, if $W(t)$ is the position of the walk after $t$ steps, we have $W(0) = 0$ and $W(t+1) = W(t) \pm 1$. Show that during the first $n$ steps, with probability at least $1 - 1/n$, the walk never ends further than $O(\sqrt{n \log n})$ from where it started, i.e., for all $t \leq n$, we have $|W(t)| = O(\sqrt{n \log n})$ with probability at least $1 - 1/n$.

(a) Express the position $W(t)$ of the walk after $t$ steps as a sum of $t$ independent random variables.

(b) Develop a Chernoff bound as in the lecture, to upper bound the probability that $W(t) \geq d$ for some $d \geq 0$.

*Hint: In order to upper bound* $\mathbb{P}\big(W(t) \geq d\big)$, *you can use that for all* $x \in \mathbb{R}$: $(e^x + e^{-x})/2 \leq e^{x^2/2}$.

(c) Use the derived bound to show the claim about $|W(t)|$ for $t \leq n$.

*Remark: If you did not succeed in (b), you can also use the Chernoff bound from the lecture to prove that* $|W(t)| = O(\sqrt{n \log n})$ *with probability at least* $1 - 1/n$ *for all* $t \leq n$.

(d) Let us now consider a random walk on the $d$-dimensional mesh. The walk starts at the origin $(0, \dots, 0)$ and in each step, it picks a uniformly random one of the $d$ dimensions and walks one step in the positive or in the negative direction in that dimension (each with probability 1/2). Show that after $n$ steps, the Euclidean distance to the origin is at most $O(\sqrt{n \log n})$ with probability at least $1 - 1/n$. Note that your argument should work even if $d \in o(n/\ln n)$ is super-constant.

## Sample Solution

(a) Let $X_s \in \{-1, 1\}$ be the direction that is taken in the $s^{\text{th}}$ step (the $X_s$ are independent from what we know from the exercise statement). Then $W(t) = \sum_{s=1}^{t} X_s$.

(b) Let $X = \sum_{i=1}^{n} X_i$ for independent random variables $X_i \in \{-1, 1\}$ with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = 1/2$ (with respect to the exercise have $X = W(n)$).

We upper bound the probability $P\big(X \geq k\big)$ for any $t > 0$ as follows

$$\mathbb{P}\big(X \geq k\big) = \mathbb{P}\big(e^{tX} \geq e^{tk}\big) \overset{\text{Markov}}{\leq} \frac{\mathbb{E}(e^{tX})}{e^{tk}}$$

We take a closer look at the expectation of $e^{tX}$:

$$\begin{aligned}
\mathbb{E}\big(e^{tX}\big) &= \mathbb{E}\Big(\prod_{i=1}^{n} e^{tX_i}\Big) \overset{X_i \text{ ind.}}{=} \prod_{i=1}^{n} \mathbb{E}\big(e^{tX_i}\big) \\
&= \prod_{i=1}^{n} \frac{1}{2}\big(e^{-t} + e^{t}\big) \overset{\text{Hint}}{\leq} \prod_{i=1}^{n} e^{t^2/2} = e^{nt^2/2}
\end{aligned}$$

We continue with the inequality from before

$$\mathbb{P}\big(X \geq k\big) \leq \frac{\mathbb{E}(e^{tX})}{e^{tk}} \leq e^{\frac{nt^2}{2} - tk}$$

We find that $t = \frac{k}{n}$ minimizes the probability and hence $\mathbb{P}\big(X \geq k\big) \leq e^{-\frac{k^2}{2n}}$.

(c) We want to give an upper bound for $\mathbb{P}\big(|X| \geq k\big)$. Since $\mathbb{P}\big(X \geq k\big) = \mathbb{P}\big(X \leq -k\big)$ due to symmetry, we obtain with a union bound:

$$\mathbb{P}\big(|X| \geq k\big) = \mathbb{P}\big(X \geq k \cup X \leq -k\big) \leq 2\mathbb{P}\big(X \geq k\big) \overset{\text{(b)}}{\leq} 2e^{-\frac{k^2}{2n}}.$$

This means we have

$$\mathbb{P}\Big(|W(n)| \geq \sqrt{cn \log n}\Big) \leq 2e^{-\frac{cn \log n}{2n}} = \frac{2}{n^{c/2}} \leq \frac{1}{n^{c'}},$$

for $c \geq 2(c' + \log_n 2) \in O(1)$.

(d) Let $Y_j = \sum_{i=1}^n Y_{ij}$ with $\mathbb{P}\big(Y_{ij} = 1\big) = \frac{1}{d}$ and $\mathbb{P}\big(Y_{ij} = 0\big) = \frac{d-1}{d}$ be the number of steps we take in dimension $j \in \{1, \ldots, d\}$ (in positive or negative direction). Given that we take $n$ steps in total, we have $\mathbb{E}(Y_j) = \frac{n}{d}$. We assume that the dimension can be super constant but not too large: $d \leq n/(3c \ln n)$ for constant $c > 0$. We analyze the probability of $Y_j \geq 2\mathbb{E}(Y_j)$ with a Chernoff bound

$$\mathbb{P}\big(Y_j \geq 2\mathbb{E}(Y_j)\big) \leq e^{-n/(3d)} \leq e^{c \ln n} = \frac{1}{n^c}.$$

Under the assumption that $Y_j \leq \frac{2n}{d}$ we have that for the distance $W_j(n)$ in dimension $j$ after $n$ steps of the "mesh-walk": $|W_j\big(n\big)| \leq |W\big(\frac{2n}{d}\big)| \leq \sqrt{2c'n/d \cdot \log(2n/d)}$ w.h.p. for some constant $c' > 0$, according to part (c). Then, assuming the above is true for each dimension $j \in \{1, \ldots, d\}$ we have for the euclidean distance from the origin in the mesh:

$$\left(\sum_{j=1}^{d} W_j\Big(\frac{2n}{d}\Big)^2\right)^{1/2} \leq \left(\sum_{j=1}^{d} \frac{2c'n}{d} \cdot \log\Big(\frac{2n}{d}\Big)\right)^{1/2} = \Big(2c'n \cdot \log\Big(\frac{2n}{d}\Big)\Big)^{1/2} \in O\Big(\sqrt{n \log(n/d)}\Big).$$

We union bound over all of the above events for each dimension to obtain that the desired outcome occurs w.h.p. (c.f. lemma on the intersection of events occuring w.h.p., which we proved in the solution of exercise 1 problem set 1).

## Exercise 2: Graph Connectivity

Let $G = (V, E)$ be a graph with $n$ nodes and edge connectivity[1] $\lambda \geq \frac{16 \ln n}{\varepsilon^2}$ (where $0 < \varepsilon < 1$). Now every edge of $G$ is removed with probability $\frac{1}{2}$. We want to show that the resulting graph $G' = (V, E')$ has connectivity $\lambda' \geq \frac{\lambda}{2}(1 - \varepsilon)$ with probability at least $1 - \frac{1}{n}$. This exercise will guide you to this result.
*Remark: If you don't succeed in a step you can use the result as a black box for the next step.*

(a) Assume you have a cut of $G$ with size $k \geq \lambda$. Show that the probability that the same cut in $G'$ has size *strictly smaller* than $\frac{k}{2}(1 - \varepsilon)$ is at most $e^{-\frac{\varepsilon^2 k}{4}}$.

(b) Let $k \geq \lambda$ be fixed. Show that the probability that at least one cut of $G$ with size $k$ becomes a cut of size *strictly smaller* than $\frac{k}{2}(1 - \varepsilon)$ in $G'$ is at most $e^{-\frac{\varepsilon^2 k}{8}}$.
*Hint: You can use that for every $\alpha \geq 1$, the number of cuts of size at most $\alpha\lambda$ is at most $n^{2\alpha}$.*

(c) Show that for large $n$ the probability that at least one cut of $G$ with *any* size $k \geq \lambda$ becomes a cut of size *strictly smaller* than $\frac{k}{2}(1 - \varepsilon)$ in $G'$, is at most $\frac{1}{n}$.
*Hint: Use another union bound.*

[1] The connectivity of a graph is the size of the smallest cut $(S, V \setminus S)$ in $G$.

## Sample Solution

(a) Let $C$ be the edges of a cut of size $k$. For $e \in C$ let $X_e = 1$ if $e \in G'$ and else $X_e = 0$. Let $X = \sum_{e\in C} X_e$. The expectation is $\mathbb{E}[X] = \sum_{e\in C} \mathbb{E}[X_e] = \frac{k}{2}$. We use a Chernoff bound

$$\Pr\big(X < \mathbb{E}[X](1-\varepsilon)\big) \leq \exp\Big(-\frac{\varepsilon^2 \mathbb{E}[X]}{2}\Big) = \exp\Big(-\frac{\varepsilon^2 k}{4}\Big).$$

(b) According to the hint we have at most $n^{\frac{2k}{\lambda}}$ many cuts of size $k$. For an arbitrary cut of $C$ of size $k$ let $C' := C \cap E'$. With a union bound we obtain

$$\begin{aligned}
\Pr\Big(\bigcup_{\substack{C \text{ cut}\\ |C|=k}} \big(|C'| < \frac{k}{2}(1-\varepsilon)\big)\Big) &\leq \sum_{\substack{C \text{ cut}\\ |C|=k}} \Pr\Big(|C'| < \frac{k}{2}(1-\varepsilon)\Big) && \text{(union bound)}\\
&\leq \sum_{\substack{C \text{ cut}\\ |C|=k}} \exp\Big(-\frac{\varepsilon^2 k}{4}\Big) && \text{(a)}\\
&\leq n^{\frac{2k}{\lambda}} \exp\Big(-\frac{\varepsilon^2 k}{4}\Big) && \text{(Hint)}\\
&= \exp\Big(\frac{2k\ln n}{\lambda}\Big) \exp\Big(-\frac{\varepsilon^2 k}{4}\Big) && (\lambda \geq \tfrac{16\ln n}{\varepsilon^2})\\
&\leq \exp\Big(\frac{\varepsilon^2 k}{8} - \frac{\varepsilon^2 k}{4}\Big) = \exp\Big(-\frac{\varepsilon^2 k}{8}\Big).
\end{aligned}$$

(c) For brevity let $\mathcal{E}(k)$ (for $k \geq \lambda$) be the event $\bigcup_{\substack{C \text{ cut}\\ |C|=k}} \big(|C'| < \frac{k}{2}(1-\varepsilon)\big)$ (recall $C' := C \cap G'$). Then

$$\begin{aligned}
\Pr\Big(\bigcup_{k=\lambda}^{n-1} \mathcal{E}(k)\Big) &\leq \sum_{k=\lambda}^{n-1} \Pr\big(\mathcal{E}(k)\big) \overset{(b)}{\leq} \sum_{k=\lambda}^{n-1} \exp\Big(-\frac{\varepsilon^2 k}{8}\Big) \leq \sum_{k=\lambda}^{n-1} \exp\Big(-\frac{\varepsilon^2 \lambda}{8}\Big)\\
&\leq \sum_{k=\lambda}^{n-1} \exp\big(-2\ln n\big) = \sum_{k=\lambda}^{n-1} n^{-2} \leq \frac{1}{n}. && (\lambda \geq \tfrac{16\ln n}{\varepsilon^2})
\end{aligned}$$

Or another solution with a geometric series which holds for large $n$

$$\begin{aligned}
\Pr\Big(\bigcup_{k=\lambda}^{n-1} \mathcal{E}(k)\Big) &\leq \Pr\Big(\bigcup_{k=\lambda}^{\infty} \mathcal{E}(k)\Big) \leq \sum_{k=\lambda}^{\infty} \Pr\big(\mathcal{E}(k)\big) \overset{(b)}{\leq} \sum_{k=\lambda}^{\infty} \exp\Big(-\frac{\varepsilon^2 k}{8}\Big) && \text{(geometric series)}\\
&= \frac{e^{-\frac{\varepsilon^2\lambda}{8}}}{1-e^{-\frac{\varepsilon^2}{8}}} = \frac{n^{-2}}{1-e^{-\frac{\varepsilon^2}{8}}} = \frac{1}{n} \cdot \underbrace{\frac{1}{n\cdot(1-e^{-\frac{\varepsilon^2}{8}})}}_{\text{fraction } \leq 1 \text{ for large } n} \leq \frac{1}{n}.
\end{aligned}$$