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# Advanced Algorithms
# Sample Solution Problem Set 3

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## Exercise 1: Tree with Small Average Stretch

Let $G = (V, E)$ with a distance metric $d_G$. Moreover, let $w : V^2 \to \mathbb{R}_{\geq 0}$ be a weight function on pairs of nodes. A tree $T$ has average stretch $\alpha$ if

(1.) $\forall u, v \in V : \ d_T(u, v) \geq d_G(u, v)$

(2.) $\sum_{u,v \in V} w(u, v) d_T(u, v) \leq \alpha \cdot \sum_{u,v \in V} w(u, v) d_G(u, v)$.

Show that, given a probabilistic tree embedding $\mathcal{T}$ with stretch $\alpha \in O(\log n)$, you can obtain a tree with average stretch $\alpha$ w.h.p.

## Sample Solution

We know that condition (1.) is fulfilled by all trees of the probabilistic tree embedding. Furthermore we know that

$$\forall u, v \in V : \mathbb{E}(d_T(u, v)) \leq \alpha \cdot d_G(u, v),$$

for some $\alpha \in O(\log n)$. This implies

$$\sum_{u,v \in V} w(u, v) \mathbb{E}(d_T(u, v)) \leq \alpha \sum_{u,v \in V} w(u, v) d_G(u, v).$$

Due to linearity of expectation this is equivalent to

$$\mathbb{E}\Big( \sum_{u,v \in V} w(u, v) d_T(u, v) \Big) \leq \alpha \sum_{u,v \in V} w(u, v) d_G(u, v).$$

Let $S_T := \sum_{u,v \in V} w(u, v) d_T(u, v)$ and $S_G := \sum_{u,v \in V} w(u, v) d_T(u, v)$, i.e. we have $\mathbb{E}\big(S_T\big) \leq \alpha S_G$. Then due to the Markov inequality

$$\mathbb{P}\big(S_T \geq 2\alpha S_G\big) \leq \mathbb{P}\big(S_T \geq 2\mathbb{E}(S_T)\big) \leq \frac{1}{2}.$$

If we sample $c \log n$ random trees $\mathcal{S} \subseteq \mathcal{T}$ from the probabilistic tree embedding $\mathcal{T}$ and we have

$$\mathbb{P}\big(\forall T \in \mathcal{S} : S_T \geq 2\alpha S_G\big) \leq \Big(\frac{1}{2}\Big)^{c \log n} = \frac{1}{n^c}.$$

That means that w.h.p. we have at least one tree in $\mathcal{S}$ with $S_T \leq 2\alpha S_G$, thus fulfilling condition (2.).

## Exercise 2: Computing Steiner Forests

Let $G = (V, E)$ with edge weights $w : E \to \mathbb{R}_{\geq 0}$. Furthermore let $\{s_1, t_1\}, \ldots, \{s_k, t_k\} \in \binom{V}{2}$ be a set of pairs of *terminals*. In the Steiner forest problem we are asking for a subset $E' \subseteq E$ with minimal weight $w(E') := \sum_{e \in E'} w(e)$, such that in $G[E']$ each pair $s_i, t_i$ is connected. Use the FRT-algorithm to compute an $O(\log n)$ approximation of a minimal weight Steiner forest $E'$ w.h.p.

*Hint: Sample a tree $T$ from a probabilistic tree embedding of $G$, solve the problem on $T$, extract a solution for $G$ and compare the result to an optimal solution for $G$.*

## Sample Solution

We use the FRT algorithm presented in the lecture to realize the probabilistic tree embedding. Then we sample a single tree $T$. That tree fulfills the following conditions:

(1.) $\forall u, v \in V : \; d_T(u, v) \geq d_G(u, v)$

(2.) $\forall u, v \in V : \; \mathbb{E}\big(d_T(u, v)\big) \leq \alpha \cdot d_G(u, v), \alpha \in O(\log n)$.

The tree $T$ still contains artificial nodes and edges from the FRT construction. We can get rid of the artificial nodes as follows. As long as we have an edge $\{u, w\}$ in $T$, where $v \in V$ and $w$ is an artificial node (i.e., one of the root nodes from our construction), we contract the edge $\{u, w\}$ and identify the new node with the actual node $u$. This merges the artificial nodes bottom up with our actual nodes from $V$. The result is a tree $T'$ on $V$ (but with "artificial" edges that are not in $G$).
However, due to the edge contractions condition (1.) given above might be violated in $T'$. We remedy this by multiplying all edges of $T'$ by a factor 4. It is clear, that the edge-contraction and multiplying tree edges by a factor of 4, condition (2.) remains valid for $T'$ with a factor $4\alpha \in O(\log n)$:

$$\mathbb{E}\big(d_{T'}(u, v)\big) \leq 4\alpha \cdot d_G(u, v). \tag{1}$$

We show that condition (1.) applies as well. Let $u, v \in V$ and let $w$ be the closest common ancestor of $u$ and $v$. Assume $w$ is at level $i$ of our FRT construction. Let $u'$ and $v'$ the ancestors of $u, v$ in $T$ directly below $w$. Then according to our contraction strategy either $\{u'w\}$ or $\{v', w\}$ in $T$ get contracted, but not both. Thus either the edge $\{u', w\}$ or $\{v', w\}$ remains in $T'$, both of which have length $D_i := D/2^i$ (before multiplication by 4). Hence $d_{T'}(u, v) \geq 4 \cdot D_i$. By FRT-construction of $T$ we have

$$d_G(u, v) \leq d_T(u, v) = d_T(u, w) + d_T(w, v) \leq 2(D_i + D_{i+1} + ... + D_{\lfloor \log_2 D \rfloor}) \leq 4D_i \leq d_{T'}(u, v). \tag{2}$$

Now we project edges of $T'$ to paths in $G$ to obtain an approximate solution $E' \subseteq E$ of a minimum Steiner forest of $G$. But first, in order to compare $E'$ to the optimal Steiner tree $E^*$ of $G$ we introduce some notions.
For each $\{s_i, t_i\}$ let $P_i^T$ and $P_i^G$ be fixed shortest paths in $T'$ and $G$ respectively. For each pair $u, v \in V$ let $P_{uv}^T$ and $P_{uv}^G$ be fixed shortest paths from $u$ to $v$ in $T'$ and $G$ respectively. We define

$$E'_T := \bigcup_{i=1}^{k} P_i^T, \quad E' := \bigcup_{\{u,v\} \in E'_T} P_{uv}^G, \quad E^*_T := \bigcup_{\{u,v\} \in E^*} P_{uv}^T.$$

Note that $E'_T$ is the optimal Steiner tree on $T'$. Also note that $E'$ can be computed in polynomial time by finding shortest paths in $G, T'$. The edge sets $E'_T$ and $E^*_T$ can be seen as projections of the approximate and optimal solutions $E'$ and $E^*$ on $G$ back to $T'$. Then we have

$$w(E') = \sum_{\{u,v\} \in E'} d_G(u, v) \overset{\text{Eq.(2)}}{\leq} \sum_{\{u,v\} \in E'} d_{T'}(u, v) = w(E'_T) \overset{E'_T \text{ opt. on } T'}{\leq} w(E^*_T) = \sum_{\{u,v\} \in E^*} d_{T'}(u, v)$$

Then we learn from condition (2.)

$$\mathbb{E}\big(w(E')\big) \leq \mathbb{E}\Big(\sum_{\{u,v\} \in E^*} d_{T'}(u, v)\Big) = \sum_{\{u,v\} \in E^*} \mathbb{E}\big(d_{T'}(u, v)\big) \overset{\text{Eq.(1)}}{\leq} 4\alpha \cdot \sum_{\{u,v\} \in E^*} d_G(u, v) = 4\alpha \cdot w(E^*).$$

We obtain a result w.h.p., that is at most twice the expectation as described in Exercise 1.