University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
P. Schneider

Advanced Algorithms
Sample Solution Problem Set 9

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Exercise 1: Counting Cuts

It is known that a graph with edge connectivity at least λ contains at least $\lambda/2$ edge disjoint spanning trees (a result from Tutte and Nash-Williams from the 1960s). Use this result to show that there are at most $O(\lambda n^{2\alpha})$ cuts of size at most $\alpha\lambda$.

Sample Solution

Let G be a graph with edge connectivity λ and let $T_1, \dots, T_k$ edge disjoint trees of G with $k \geq \lambda/2$. Let $(S, V \setminus S)$ be a cut of size at most $\alpha\lambda$, i.e. the number of edges $E(S)$ crossing the cut is at most $\alpha\lambda$. There must be a tree T_i with $|E(S) \cap T_i| \leq 2\alpha$, otherwise the cut would have

$$|E(S)| = \sum_{i=1}^{k} |E(S) \cap T_i| > \sum_{i=1}^{k} 2\alpha \geq \alpha\lambda$$

many edges, a contradiction. Since the T_i are spanning trees every tree must have a non-empty intersection with S. Hence for each cut there is a tree T_i with

$$1 \leq |E(S) \cap T_i| \leq 2\alpha.$$

Let S and S' be two different cuts (i.e. $S \neq S'$ and $E \setminus S \neq S'$). Then we have the following

(i) The number of subsets of T_i of size at most 2α is $O(n^{2\alpha})$,

(ii) $E(S) \cap T_i$ and $E(S') \cap T_i$ are two different sets.

We prove this separately further below. From (i) and (ii) it follows, that each T_i can play the role of having at most 2α edges in common with some cut of size $\lambda\alpha$ at most $O(n^{2\alpha})$ times. Thus we can have at most $O(\lambda n^{2\alpha})$ of these cuts.

We prove (i). Each T_i has exactly $n-1$ edges (since these are trees). The number of subsets of T_i of size at most 2α is

$$\sum_{i=1}^{\lfloor 2\alpha \rfloor} \binom{n-1}{i} \leq \sum_{i=1}^{\lfloor 2\alpha \rfloor} (n-1)^i \overset{(*)}{=} \frac{(n-1)^{\lfloor 2\alpha \rfloor + 1}}{n-1-1} - 1 \in O(n^{2\alpha})$$

The step $(*)$ is the partial sum of the geometric series $\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$.

It remains to show (ii), i.e., that $E(S) \cap T_i$ and $E(S') \cap T_i$ are two different sets. Let $u \in S$ with $u \notin S'$. Let $v \in E \setminus S$ with $v \notin S'$. Both nodes exist due to S and S' being different cuts. It is $\{u, v\} \in E(S)$ but $\{u, v\} \notin E(S')$. Assume there are u, v as described before such that $\{u, v\} \in T_i$, then we are done. Otherwise, any u and v with the described properties must be connected via some path P through T_i. Since exists no pair $\{x, y\} \in T_i$ with $x \in S \setminus S'$ and $y \in (E \setminus S) \setminus S'$, this path P must first lead into the set $(E \setminus S) \cap S'$ and then into $(E \setminus S) \setminus S'$. Therefore T_i has an edge $\{u', v'\}$ over the cut S' with $u' \in (E \setminus S) \cap S'$ and $v' \in (E \setminus S) \setminus S'$. Hence $\{u', v'\} \in E(S')$ but $\{u', v'\} \notin E(S')$.

Exercise 2: Approximating Cuts in Graphs with Large Expansion

Let $G = (V, E)$ be an unweighted graph for which the following property holds. For all cuts S (w.l.o.g. we assume $|S| \leq |E \setminus S|$ otherwise we switch the roles of S and $E \setminus S$) and some (large) constant α we have that $e(S)/|S| \geq \alpha$. Show that by sampling edges with probability $p := \min(\frac{c \ln n}{\alpha\varepsilon^2}, 1)$ and assigning appropriate weights, w.h.p. we obtain a subgraph with $\tilde{O}(|E|/\alpha)$ edges that is an $(1 \pm \varepsilon)$-approximation of all cuts (for constant $\varepsilon > 0$).

Sample Solution

Let G' be the graph after sampling edges and let S be a cut. We assume that $p < 1$, otherwise $G = G'$ and we are done. Let us call $e'(S)$ the size of the cut S in G'. The expectation of $e'(S)$ is

$$\mathbb{E}\big(e'(S)\big) = p \cdot e(S) = \frac{e(S) \cdot c \ln n}{\alpha\varepsilon^2}.$$

Let us consider the probability that $e'(S)$ deviates more than a factor of $(1+\varepsilon)$ from its expectation:

$$\begin{aligned}
\mathbb{P}\Big(|e'(S) - \mathbb{E}\big(e'(S)\big)| \geq \varepsilon \mathbb{E}\big(e'(S)\big)\Big) &\leq 2 \exp\Big(-\frac{\varepsilon^2 \mathbb{E}\big(e'(S)\big)}{3}\Big) \\
&\leq 2 \exp\Big(-\frac{\varepsilon^2 e(S) \cdot c \ln n}{3\alpha\varepsilon^2}\Big) \\
&\overset{c' := c/3}{=} 2 \exp\Big(\frac{e(S) c' \ln n}{\alpha}\Big) \\
&\leq 2 \exp\big(|S| c' \ln n\big) \\
&= 2n^{-c'|S|}
\end{aligned}$$

Let us define the event $\overline{\mathcal{E}}(S) := |e'(S) - \mathbb{E}\big(e'(S)\big)| \geq \varepsilon \mathbb{E}\big(e'(S)\big)$. The number of cuts with size exactly k is at most n^k. With a union bound we have

$$\mathbb{P}\Big(\bigcup_{\substack{S \subseteq V \\ 1 \leq |S| \leq n/2}} \overline{\mathcal{E}}(S)\Big) \leq \sum_{k=1}^{\lfloor n/2 \rfloor} n^k \cdot \mathbb{P}\big(\overline{\mathcal{E}}(S)\big) = \sum_{k=1}^{\lfloor n/2 \rfloor} n^k \cdot 2n^{-c'k} = 2\lfloor n/2 \rfloor \cdot n^{-c'} \leq n^{-(c'-1)}.$$

Thus $\mathcal{E}(S)$ occurs w.h.p. for all cuts S. Hence the size of the cut S in G' is in the range from $(1-\varepsilon)pe(S)$ to $(1+\varepsilon)pe(S)$ w.h.p. By scaling edge weights with factor $1/p$ we have that $w\big(E'(S)\big)$ is in the range from $(1-\varepsilon)e(S)$ to $(1+\varepsilon)e(S)$ w.h.p.