# Chapter 2
# Broadcast, Convergecast, and Spanning Trees

## Distributed Systems

## SS 2019


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# Message Passing in Arbitrary Topologies

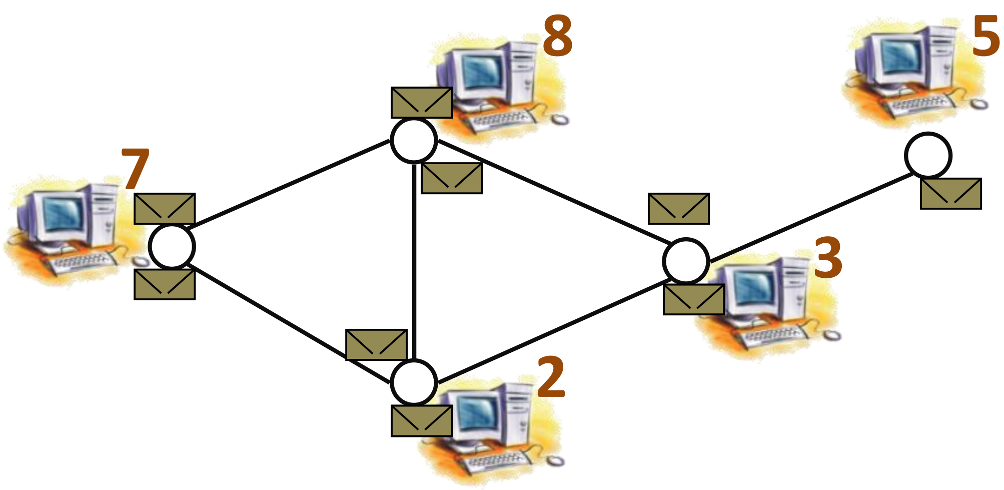
**Assumption for this chapter:**

- Network: message passing system with arbitrary topology
- network topology is given by an undirected graph $G = (V, E)$

# Synchronous Message Passing

- Time is divided into synchronous rounds

**In each synchronous round:**
1. Each node does some internal computation
2. Send a message to each neighbor
3. Receive message from each neighbor




**time complexity = number of rounds**

# Asynchronous Message Passing

**In this chapter:** No failures, but asynchrony

**Asynchronous message passing:**

- messages are always delivered in finite time
  - cf.: finite time → admissible schedule
- message delays are completely unpredictable
- algorithms are **event-based**:

**upon receiving message from neighbor ..., do**

**some local computation steps**

**send message(s) to neighbor(s) ...**

# Broadcast

- Simple, basic communication problem

**Problem Description:**

- A source node $s$ needs to broadcast a message $M$ to all nodes of the system (network)
- Each node has a unique ID
- Initially, each node knows the IDs of its neighbors
  - or it can count / distinguish its neighbors by individual communication ports to the pairwise communication links

# Flooding

- One of the simplest distributed (network) algorithms

**Basic idea:**

- When receiving $M$ for the first time, forward to all neighbors

**Algorithm:**

- Source node $s$:
  **initially do**
  send $M$ to all neighbors

- All other nodes $u$:
  **upon receiving $M$ from some neighbor $v$**
  **if** $M$ has not been received before **then**
  send $M$ to all neighbors except $v$

# Flooding in Synchronous Systems

**Synchronous systems:**

- time divided into synchronous rounds, msg. delay $= 1$ round
- time complexity: number of rounds

**Progress in flooding algorithm:**

# Flooding in Synchronous Systems

**Synchronous systems:**

- time divided into synchronous rounds, msg. delay $= 1$ round
- time complexity: number of rounds

**Progress in flooding algorithm:**

- after 1 round:
  - all neighbors of $s$ know $M$
  - nodes at distance $\geq 2$ from $s$ do not know $M$
- after 2 rounds:
  - exactly nodes at distance $\leq 2$ from $s$ know $M$
- ...
- after $r$ rounds:
  - exactly nodes at distance $\leq r$ from $s$ know $M$

# Flooding in Synchronous Systems

**Radius:** (Graph $G = (V, E)$)

- Given a node $s \in V$, radius of $s$ in $G$:

$$rad(G, s) := \max_{v \in V} dist_G(s, v)$$

- radius of $G$:

$$rad(G) := \min_{s \in V} rad(G, s)$$

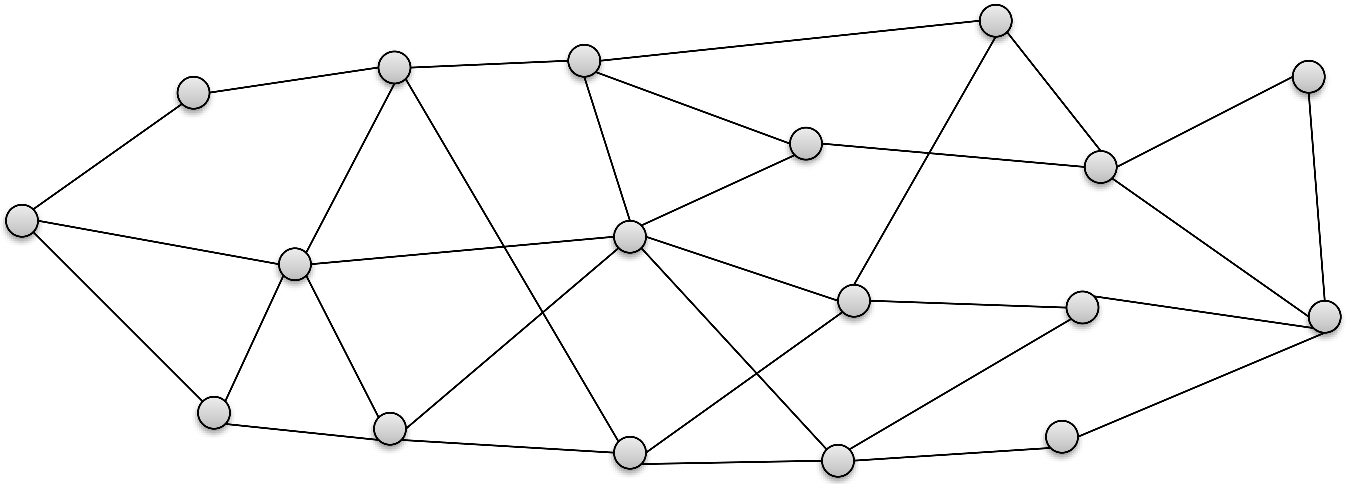
**Diameter of $G$:**

$$diam(G) := \max_{u,v \in V} dist_G(u, v) = \max_{s \in V} rad(G, s)$$

**Time complexity** of **flooding** in synchronous systems: $\boldsymbol{rad(G, s)}$

$$\frac{diam(G)}{2} \leq rad(G) \leq rad(G, s) \leq diam(G)$$

# Radius and Diameter



# Asynchronous Time Complexity

- Time complexity of flooding in asynchronous systems?
- How is time complexity in asynchronous systems defined?

**Assumptions:**

- Message delays, time for local computations are arbitrary
  - Algorithms cannot use any timing assumptions!
- **For analysis:**
  - message delays $\leq 1$ time unit
  - local computations take $0$ time

**Determine asynchronous time complexity:**

1. determine running time of a given execution
2. **asynch. time complexity $=$ max. running time of any exec.**

# Asynchronous Time Complexity

**Running time of an execution:**

- assign times to send and receive events such that
  - order of all events remains unchanged
  - local computations take $0$ time
  - message delays are at most $1$ time unit
  - first send event is at time $0$
  - time of last event is maximized

- essentially: normalize message delays such that the maximum delay is 1 time unit

**Definition Asynchronous Time Complexity:**

**Total time of a worst-case execution in which local computations take time $0$ and all message delays are at most $1$ time unit.**

# Flooding in Asynchronous Systems

**Theorem:** The time complexity of flooding from a source $s$ in an asynchronous network $G$ is $rad(G, s)$.

# Message Complexity

**Message Complexity:** Total number of messages sent (over all nodes)

What is the message complexity of flooding?

**Theorem:** The message complexity of flooding is $O(|E|)$.

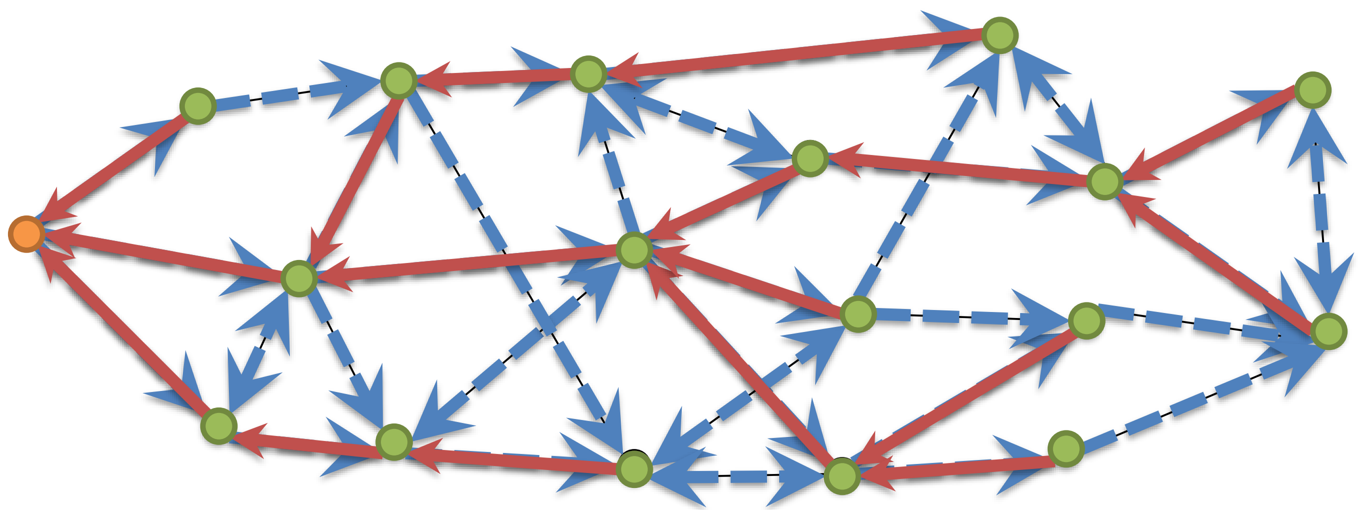
- on graph $G = (V, E)$

# Flooding Spanning Tree

- The flooding algorithm can be used to compute a spanning tree of the network.

**Idea:**

- Source $s$ is the root of the tree
- For all other nodes, neighbor from which $M$ is received first is the parent node.



# Flooding Spanning Tree Algorithm

**Source node $s$:**

**initially do**

parent $:= \bot$ // $s$ is the root

send $M$ to all neighbors

**Non-source node $u$:**

**upon receiving $M$ from some neighbor $v$**

**if** $M$ has not been received before **then**

parent $:= v$

send $M$ to all neighbors except $v$

# Spanning Tree: Synchronous Systems

- In tree: distance of $v$ to root $=$ round in which $v$ is reached
- In synchronous systems, a node $v$ are reached in round $r$ if and only if $dist_G(s, v) = r$

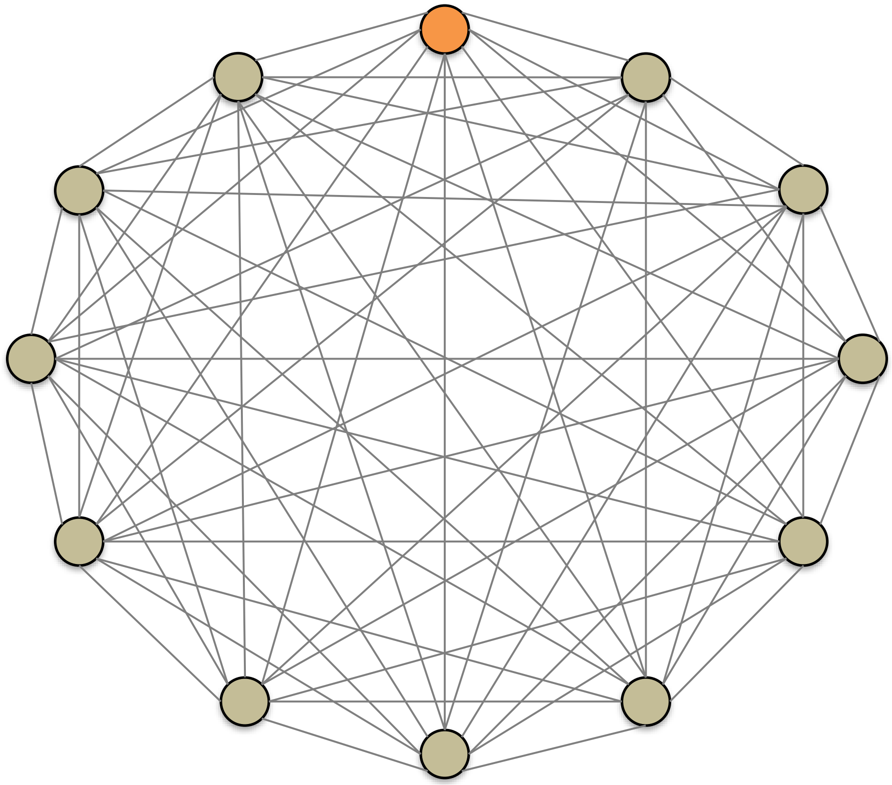
**Shortest Path Tree = BFS Tree** (BFS = breadth first search)

- tree which preserves graph distances to root node

**Theorem:** In synchronous systems, the flooding algorithm constructs a BFS tree.

# Spanning Tree: Asynchronous Systems

How does the spanning tree look if comm. is asynchronous?

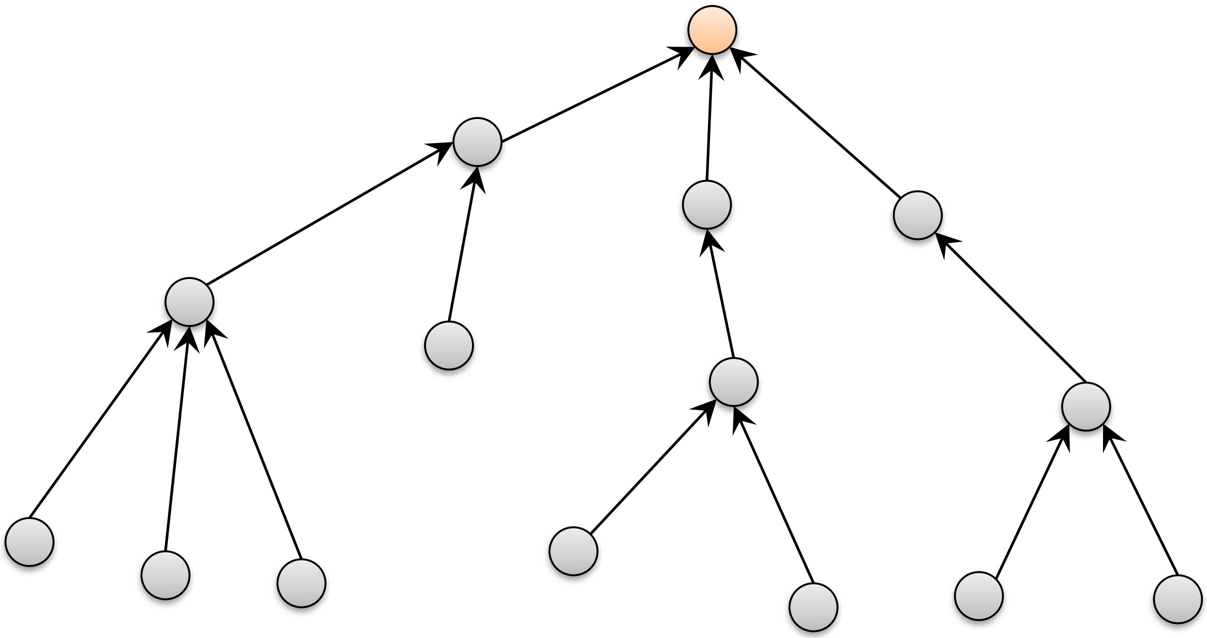


**Observation:** In asynchronous executions, the depth of the tree can be $n-1$ even if the radius/diameter of the graph is 1.

# Convergecast

- “Opposite” of broadcast
- Given a rooted spanning tree, communicate from all nodes to the root
  - starting from the leaves

Example: Compute sum of values in a rooted tree



# Convergecast Algorithm

**Leaf node $v$:**
**initially do**
send message to parent
(e.g., send input value)

**Inner node $u$:**
**upon receiving message** from child node $v$
**if** $u$ has received messages from all children **then**
send message to parent
(e.g., send sum of all inputs in $u$'s subtree)

**Root node $r$:**
**upon receiving message** from child node $v$
**if** $r$ has received messages from all children **then**
convergecast terminates

# Convergecast: Analysis & Remarks

**Time Complexity:**

**Message Complexity:**

**Application of the convergecast algorithm:**

- Computing functions, e.g.:
  - min, max, sum, average, median, ...
- Termination detection
  - inform parent as soon as all nodes in subtree have terminated
- ...

# Flooding/Echo Algorithm

- If a leader (root), but no spanning tree exists, flooding and convergecast can be used together for computing functions, ...

1. Use flooding to construct a tree
2. Use convergecast (echo) to report back to the root when done

**Time Complexity of Flooding + Convergecast (Echo):**

# Constructing Good Trees

- When combining flooding and convergecast, the time complexity is linear in the depth of the constructed tree.
- In synchronous systems, the tree is a BFS tree (shortest path tree), i.e., the depth of the tree is $O\big(diam(G)\big)$
  - optimal time complexity: $O\big(diam(G)\big)$
- In asynchronous systems, the time complexity can be $\Omega(n)$, even if the graph has a very small diameter!
- Convergecast / low diameter spanning trees are important!
- How can be construct a BFS tree in an asynchronous system?

# Constructing Shortest Path Tree

**Dijkstra**

- Grow tree from source $s$
- At intermediate step $t$, subtree of all nodes at distance $\leq r_t$ from source node $s$
- Next step: add node with min. distance to $s$

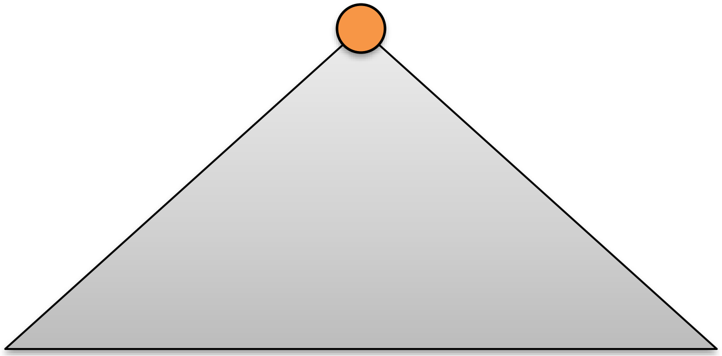
**Bellman-Ford**

- Each node $v$ keeps a distance estimate $d_v$ to $s$
  - initially: $d_s = 0,\ d_v = \infty$ (for all $v \neq s$)
- In each step, all nodes update their estimate based on neighbor estimates:

$$d_v = \min\left\{d_v, \min_{u \in N(v)}\{d_u + 1\}\right\}$$

# Distributed Dijkstra

- In our case, the graph is unweighted
- We can therefore grow the tree level by level
  - Essentially like in a synchronous execution
- Assume, the tree is constructed up to distance $r$ from $s$
- How can we add the next level?



# Distributed Dijkstra

- Source/root node coordinates the phases

**Algorithm for Phase $r + 1$:**

1. Root node broadcasts *"start phase* $r + 1$" in current tree
2. Leaf nodes (level $r$ nodes) send *"join* $r + 1$" to neighbors
3. Node $v$ receiving "*join* $r + 1$" from neighbor $u$:
    1. First such message: $u$ becomes parent of $v$, $v$ sends *ACK* to $u$
    2. Otherwise, $v$ sends *NACK* to $u$
4. After receiving *ACK* or *NACK* from all neighbors, level $r$ nodes report back to root by starting a convergecast
5. When the convergecast terminates at the root, the root can start the next phase

# Distributed Dijkstra: Analysis

**Time Complexity:**

**Message Complexity:**

# Distributed Bellman-Ford

**Basic Idea:**

- Each node $u$ stores an integer $d_u$ with the current guess for the distance to the root node $s$
- Whenever a node $u$ can improve $d_u$, $u$ informs its neighbors

**Algorithm:**

1. Initialization: $d_s \coloneqq 0$, for $v \neq s$: $d_v \coloneqq \infty$, $\text{parent}_v \coloneqq \perp$
2. Root $s$ sends "1" to all neigbors
3. For all other nodes $u$:
   **upon receiving message** "$x$" with $x < d_u$ from neighbor $v$ **do**
   set $d_u \coloneqq x$
   set $\text{parent}_u \coloneqq v$
   send "$x + 1$" to all neighbors (except $v$)

# Distr. Bellman-Ford: Time Complexity

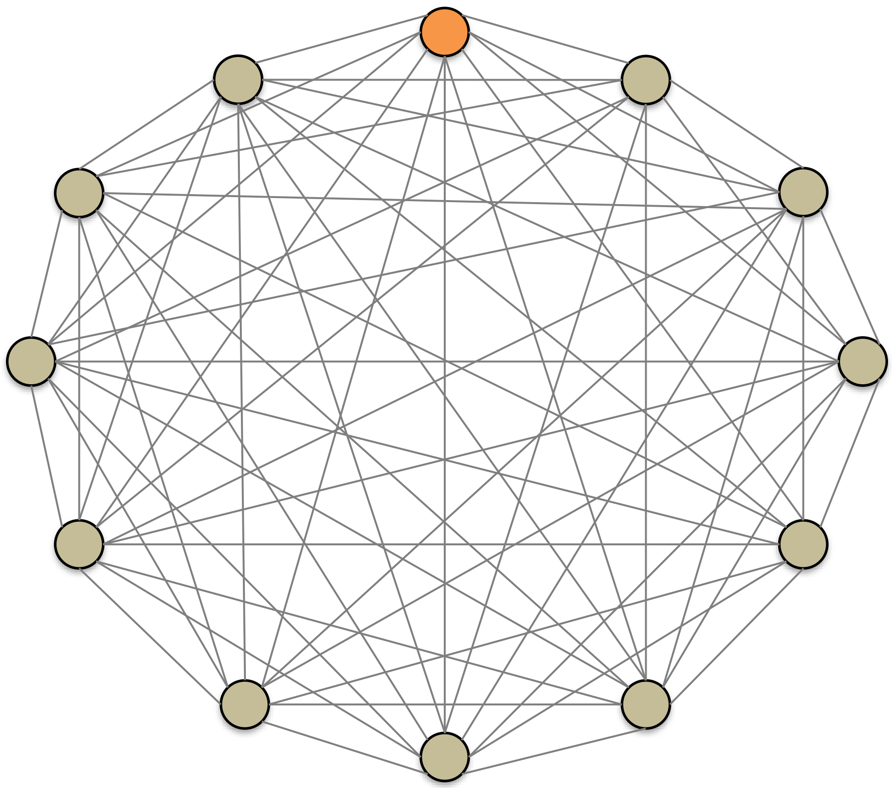
**Theorem:** The time complexity of the distributed Bellman-Ford algorithms is

# Distr. Bellman-Ford: Message Complexity

**Theorem:** The message complexity of the distributed Bellman-Ford algorithms is

# Distr. Bellman-Ford: Message Complexity

**Theorem:** The message complexity of the distributed Bellman-Ford algorithms is $O(|E| \cdot |V|)$.



# Distributed BFS Tree Construction

## Synchronous

- Time: $O(diam(G))$, Messages: $O(|E|)$
- both optimal

## Asynchronous

- **Distributed Dijkstra:**
  Time: $O(diam(G)^2)$, Messages: $O(|E| + |V| \cdot diam(G))$
- **Distributed Bellman-Ford:**
  Time: $O(diam(G))$, Messages: $O(|E| \cdot |V|)$
- **Best known trade-off between time and messages:**
  Time: $O(diam(G) \cdot \log^3|V|)$, Messages: $O(|E| + |V| \cdot \log^3|V|)$
  - based on **synchronizers**
    (generic way of translating synchronous algorithms into asynch. ones)
  - We will look at synchronizers in a later lecture…

# Leader Election

**Task:** Each node has an input value, compute sum of values

**Solution:** Compute spanning tree and use convergecast on spanning tree (i.e., flooding + convergecast)

**Problem:** What if we don't have a source/root node?

We need to choose a root node

- known as the *leader election problem*

**Solving leader election:**

- E.g.: Choose node with smallest ID
- How to find node with smallest ID?

# Solving Leader Election

**Choose node with smallest ID**

**Algorithm for node $u$:**

- Node $u$ stores smallest known ID in variable $x_u$

1. Initially, $u$ sets $x_u \coloneqq \mathrm{ID}_u$ and sends $x_u$ to all neighbors
2. when receiving $x_v < x_u$ from neighbor $v$:

   $x_u \coloneqq x_v$

   send $x_u$ to all neighbors (except $v$)

**Time Complexity:**

# Solving Leader Election

**Choose node with smallest ID**

**Algorithm for node $u$:**

- Node $u$ stores smallest known ID in variable $x_u$

1. Initially, $u$ sets $x_u \coloneqq \mathrm{ID}_u$ and sends $x_u$ to all neighbors
2. when receiving $x_v < x_u$ from neighbor $v$:

   $x_u \coloneqq x_v$

   send $x_u$ to all neighbors (except $v$)

**Message Complexity:**

# Solving Leader Election

**Choose node with smallest ID**

**Algorithm for node $u$:**

- Node $u$ stores smallest known ID in variable $x_u$

1. Initially, $u$ sets $x_u := \mathrm{ID}_u$ and sends $x_u$ to all neighbors
2. when receiving $x_v < x_u$ from neighbor $v$:
   $x_u := x_v$
   send $x_u$ to all neighbors (except $v$)

**Termination?**

# Leader Election

Simple leader election algorithm has time complexity $O\big(diam(G)\big)$ and message complexity $O(|V| \cdot |E|)$.

**Problems:**

- While time compl. is optimal, msg. complexity is extremely high
- It is not clear when/how to terminate

- Like for BFS tree construction, there are many possible trade-offs between time and message complexity, e.g.:
  - Time Complexity: $O(|V|)$, Message Complexity: $O(|E| + |V| \cdot \log|V|)$
  - Algorithm is based on constructing a spanning tree in a message-efficient way

- Termination can be solved
  - see exercises!