Maximal Independent Sets (MIS)

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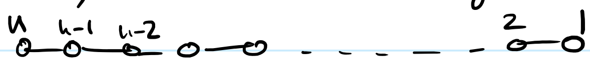
Def: Let $G=(V,E)$ be a graph. An independent set is a subsed $S \subseteq V$ of the nodes s.t. no two nodes $u,v \in S$ are neighbors in G.
An ind. set is called maximal if $\forall v \in V \setminus S$, the set $S \cup \{v\}$ is not an indep. set.

Simple Distributed MIS algorithm

- every node waits until all neighbors with higher ID are decided (v is decided = node v knows if it is in MIS)
- when all higher ID neighbors are decided, node joins MIS if possible

Thm: Simple distr. MIS alg. computes an MIS in time $O(n)$.
Proof: In each round, at least one node gets decided



Assume that a C-coloring of G is given
each node has a color $\in \{1, \dots, C\}$

Observation: With a C-coloring, an MIS can be computed in $O(C)$ rounds.
Algorithm for $x = 1$ to C do
forall nodes v of color x do (in parallel)
if v has no neighbor in MIS, v joins MIS } (*)

Claim: Alg. computes an MIS in $O(C)$ rounds
Proof:
- time complexity: (*) requires $O(1)$ rounds

1 round: every node can send a message to all neighbors and receive messages from neighbors

- set is a maximal ind. set: indep. because no 2 neighbors can join set at the same time
maximal: every node is considered

So far, we have an alg. with $O(n)$ time and one with $O(C)$ time (if a coloring is given).

How can we be faster?

Idea: use randomization

e.g., use simple dist. MIS alg. with random IDs
→ can show that this computes an MIS in $O(\log n)$ rounds.
proven by [Fischer, Noever; 2018]

We use a "slight" variation of this algorithm (by [Luby; 1986])

Algorithm consists of phases:
Initially $S = \emptyset$
In each phase:
- each undec. node v picks a random number $r_v \in [0,1]$
- if $r_v > r_u$ for all undec. neighbors u of v:
  - v joins set S
  - v informs neighbors about decision

Claim: Alg. computes an MIS.

Proof: S is an indep. set: no 2 neighbors (u and v) can join S in the same phase

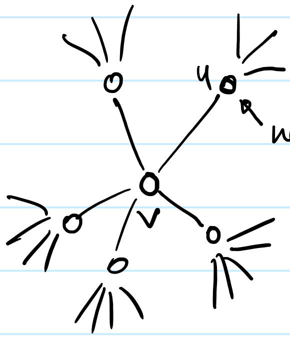
$$\left.\begin{array}{l} u \text{ joins} \rightarrow r_u > r_v \\ v \text{ join} \rightarrow r_v > r_u \end{array}\right\} \text{cannot both be true}$$

We want to show that we only need $O(\log n)$ phases.

Idea: measure progress of a single phase

We will show that (roughly), the number of edges between undecided nodes decreases by a const. factor per phase.

Which edges are removed if we add some node v to S?

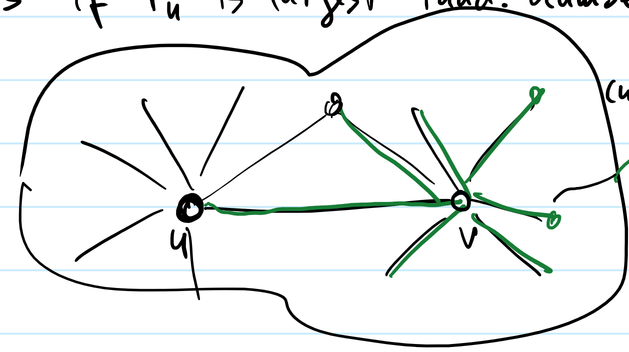


neighbor u is now also decided
(u cannot join S any more)

Assume that $G=(V,E)$ is the graph induced by undecided nodes and that $G'=(V',E')$ is the graph ind. by undec. nodes at the end of the phase.

Probability event $\mathcal{E}_{u,v}$ for all u,v s.t. $\{u,v\}\in E$

$\mathcal{E}_{u,v}$ holds if r_u is largest rand. number in $N(u)\cup N(v)$ (neighbors of u)



will be removed if $\mathcal{E}_{u,v}$ occurs

$\mathcal{E}_{u,v} \rightarrow u$ joins S

$$\mathbb{P}(\mathcal{E}_{u,v}) = \frac{1}{|N(u)\cup N(v)|} \geq \frac{1}{d(u)+d(v)}$$

$$X_{u,v} = \begin{cases} d(v) & \text{if } \mathcal{E}_{u,v} \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X_{u,v}] \geq \frac{d(v)}{d(u)+d(v)}$$

$$X = \sum_{\{u,v\}\in E} (X_{u,v} + X_{v,u}), \quad Y = |E| - |E'|$$

# of removed edges

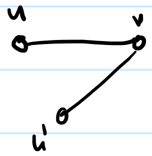
Claim: $Y \geq \frac{1}{2} \cdot X$

$\mathcal{E}_{u,v}$ $\rightarrow$ edges of v are removed

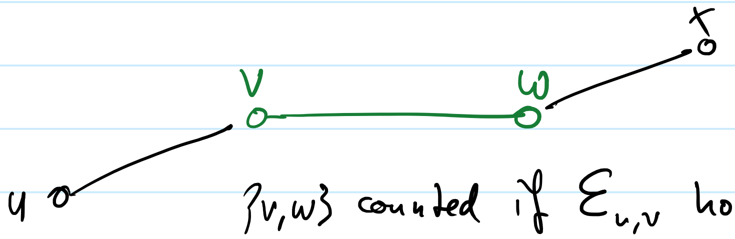
↳ contribution to X is $X_{u,v} = d(v)$

For every v, $\mathcal{E}_{u,v}$ can only occur for one node u

(not possible that $\mathcal{E}_{u,v}$ & $\mathcal{E}_{u',v}$ both occur:



$\left.\begin{array}{l} \mathcal{E}_{u,v} \rightarrow r_u > r_{u'} \\ \mathcal{E}_{u',v} \rightarrow r_{u'} > r_u \end{array}\right\}$ cannot both be true



$\{v,w\}$ counted if $\mathcal{E}_{u,v}$ holds for some $u \in N(v)$

$\{v,w\}$ " " $\mathcal{E}_{x,w}$ holds for some $x \in N(w)$

$\rightarrow$ every edge only counted $2\times$

$\Rightarrow X \leq 2 \cdot Y$ $\square$

Linearity of expectation

$$\mathbb{E}\left[\sum_{i=1}^{k} \alpha_i \cdot X_i\right] = \sum_{i=1}^{k} \alpha_i \, \mathbb{E}[X_i]$$

Lemma: $\mathbb{E}[Y] \geq \frac{1}{2} \cdot |E|$

Proof: We will show that $\mathbb{E}[X] \geq |E|$

$\Rightarrow$ Lemma then follows with previous claim

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{\{u,v\}\in E} (X_{u,v} + X_{v,u})\right]$$

$$\overset{\text{lin. of exp.}}{=} \sum_{\{u,v\}\in E} \left(\mathbb{E}[X_{u,v}] + \mathbb{E}[X_{v,u}]\right)$$

$$\geq \sum_{\{u,v\}\in E} \underbrace{\left(\frac{d(v)}{d(u)+d(v)} + \frac{d(u)}{d(u)+d(v)}\right)}_{=1}$$

$$= |E|$$

Can be extended to show that the algorithm terminates after $O(\log n)$ phases.

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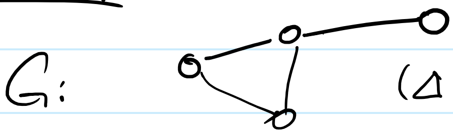
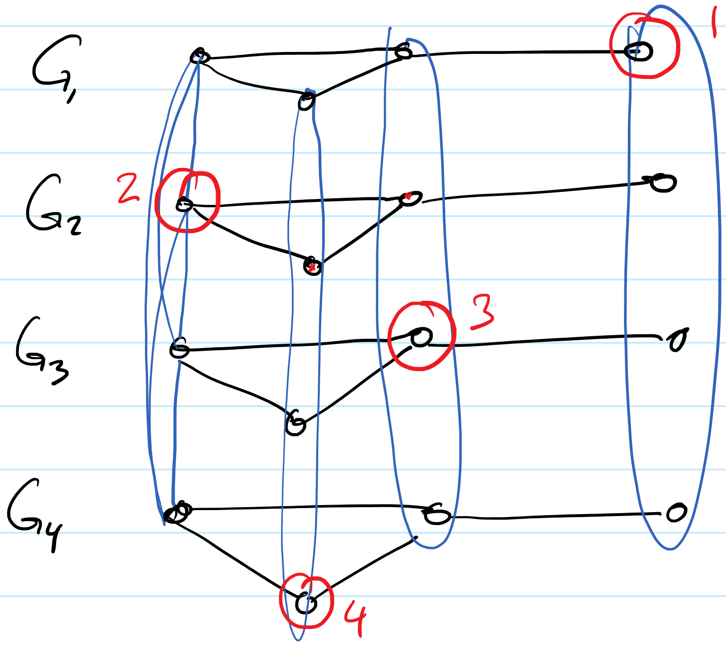
Use MIS to compute $(\Delta+1)$-coloring.

Let's look at an example



$(\Delta = 3)$

build virtual graph: (consisting of $\Delta+1$ copies of G)



$\leftarrow$ call this graph H

Alg: compute MIS of H

$\Longrightarrow$ solves $(\Delta+1)$ coloring in $O(\log n)$ rounds.