In the following exercises we consider the CONGEST model. This is a synchronous message passing model with the additional property that the size of each message is bounded. If we assume that the nodes have IDs in \( \{1, \ldots, n\} \) and communicate by exchanging bitstrings, then each message is only allowed to contain \( O(\log n) \) bits. This means that each message may contain for example (the binary representation of) a constant number of integers \( \leq n^c \) for some constant \( c \). However, it is not possible that a node sends another node the IDs of all its neighbors in a single message, as the degree of the network may not be bounded.

Remark: Do not confuse the message size and the message complexity.

1. **k-Selection Problem in Graphs**

   Given a graph \( G \) with \( n \) nodes that have pairwise distinct input values \( \leq n^c \) for some constant \( c \), the \( k \)-selection problem for a \( k \leq n \) is the problem of finding the \( k^{th} \)-smallest value in the graph.

   Our goal is to describe a randomized distributed algorithm in the CONGEST model that solves the \( k \)-selection problem with an expected runtime of \( O(D \cdot \log n) \).

   a) Assume a tree \( T \) of depth \( D \). Describe an algorithm that computes in \( O(D) \) rounds for every node \( v \) a value \( s_v \) which equals the size (number of nodes) of the subtree with root \( v \).

   b) Assume a tree \( T \) of depth \( D \) and root \( r \) in which each node is able to flip coins. Describe a method to choose a node from the tree uniformly at random (i.e., each node has the same probability to be chosen) in time \( O(D) \).

   \( \text{Hint: Use the algorithm from a).} \)

   c) Assume a tree \( T \) of depth \( D \), where each node \( v \) has in input a boolean \( b_v \). Modify the algorithm of a) such that for every node \( v \), the value \( s_v \) is equal to the number of nodes in the subtree rooted at \( v \) that have \( b = \text{True} \). Also, modify the algorithm from b) to choose uniformly at random a node among all nodes that have \( b = \text{True} \).

   d) Describe a randomized algorithm that solves the \( k \)-selection problem with an expected runtime of \( O(D \cdot \log n) \).

   \( \text{Hint: Use the algorithms from c).} \)

2. **Leader Election**

   Given a graph \( G \), describe a deterministic algorithm in the CONGEST model such that every node learns the smallest ID in the graph and terminates after \( O(D) \) rounds. Analyse the message complexity of the algorithm.