Distributed Systems, Summer Term 2020
Exercise Sheet 7

1 Maximal matching

In the following, we are given a graph $G = (V, E)$ of maximum degree $\Delta$, where nodes are colored with $c$ colors, and the goal is to produce a maximal matching. A maximal matching is a subset of edges $X \subseteq E$ satisfying the following:

- For all $e_1, e_2$ in $X$, it holds that $e_1$ and $e_2$ are not incident to the same node, that is, they do not share endpoints. Hence, for each node it holds that at most one incident edge is in the matching.
- Adding any additional edge of $E \setminus X$ to $X$ would violate the above constraint.

Hence, we are interested in a subset of edges that are independent such that this subset cannot be extended.

1. Consider the case where $c = 2$, that is, the graph is bipartite and properly colored with two colors, black and white. Assume that nodes know the value of $\Delta$ and $c$. Show that maximal matching can be solved in $O(\Delta)$ rounds. Hint: it can be solved in $2\Delta$ rounds. Spoiler hint: see the footnote \(^1\).

2. Assume that $c$ and $\Delta$ are known to each node. Show that, for any value of $c$, this problem can be solved in $O(c \Delta)$.

3. Show that this problem can be solved in $O(c \Delta)$ even in the case where $c$ and $\Delta$ are unknown to the nodes.

2 Coloring planar graphs

Show how to color a planar graph with $O(1)$ colors in $O(\log n)$ time. Hint: every planar graph satisfies that the average degree of the nodes is less than 6. Hint: use the same idea of the algorithm for unrooted trees presented in the lecture.

3 Coloring unrooted trees

Show that it is possible to 3-color unrooted trees in $O(\log n)$ time. Hint: modify the algorithm that 9-colors unrooted trees presented in the lecture.

\(^1\)Black nodes can accept the first proposal and reject all the others.
White nodes can try to "propose" to each black neighbor by trying one neighbor at a time.
4 Color Reduction

a) Given a graph which is colored with $m > \Delta + 1$ colors, describe a method to recolor the graph in one round using $m - \lfloor \frac{m}{\Delta+2} \rfloor$ colors.

*Hint: Partition the set of colors into sets of size $\Delta + 2$ and recall the color reduction method from the lecture.*

b) Show that after $O(\Delta \log(m/\Delta))$ iterations of step a), one obtains a $O(\Delta)$ coloring.