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# Distributed Systems, Summer Term 2019
# Exercise Sheet 8

In both exercises we consider the synchronous message passing model.

## 1 MIS Application: Matching

A **matching** of a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ such that no two edges in $M$ are adjacent. A matching is maximal if no edge can be added without violating this property.

Give an algorithm that computes a maximal matching in $O(\log n)$ rounds w.h.p.

## Sample Solution

Let $G = (V, E)$ be the graph for which we want to construct the matching. The so-called line graph $G'$ is defined as follows: for every edge in $G$ there is a node in $G'$; two nodes in $G'$ are connected by an edge if their respective edges in $G$ are adjacent. $G'$ can be simulated on $G$ with constant overhead. A (maximal) independent set in the line graph $G$ is a (maximal) matching in the original graph $G$, and vice versa. If $G$ has $n$ nodes, $G'$ has at most $n^2$ nodes and so we can compute an MIS on $G'$ in time $O(\log n^2) = O(\log n)$.

## 2 MIS Application: Dominating Set

A **dominating set** of a graph $G = (V, E)$ is a subset of the nodes $D \subseteq V$ such that each node is in $D$ or adjacent to a node in $D$. A minimum dominating set is a dominating set containing the least possible number of nodes.

$G = (V, E)$ has **neighborhood independence** $\beta$ if for every node $v \in V$ the largest independent set in the neighborhood $N(v)$ is of size at most $\beta$.

a) Show that for an MIS $M$ and a minimum dominating set $D$ of a graph it holds $|D| \leq |M|$.

b) Give a class of graphs each containing an independent set $I$ and a dominating set $D$ with $\frac{|I|}{|D|} = O(n)$.

c) Show that for graphs with neighborhood independence $\beta \geq 1$, a $\beta$-approximation to a minimum dominating set (that is a dominating set which is at most $\beta$ times larger than a minimum dominating set) can be found in time $O(\log n)$ w.h.p.

d) A unit disc graph is a graph $(V, E)$ with $V \subset \mathbb{R}^2$ and $E = \{\{u, v\} \mid \|u - v\|_2 \leq 1\}$. Show that one can compute a 5-approximation to a minimum dominating set in disc graphs in time $O(\log n)$ w.h.p.

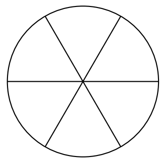
## Sample Solution

a) Notice that, by definition, every MIS is a dominating set. Suppose that $|M| > |D|$, it means that we have a dominating set that has a smaller size than the one of the minimum dominating set, which is a contradiction.

b) For every $n$, take a star graph with $n$ nodes which has a dominating set of size 1 and an independent set of size $n-1$.

c) We compute an MIS $I$ in time $O(\log n)$ which is a dominating set. We have to show that for a minimum dominating set $D$ we have $|I| \leq \beta|D|$. Each node in $I$ is either in $D$ or neighbor of a node in $D$. So for counting the nodes in $I$, we can iterate through the nodes in $D$ and count in each step the number of nodes in $I$ which are covered by the corresponding node in $D$. Each node in $D$ has at most $\beta$ neighbors in $I$. Therefore we will count at most $\beta|D|$ nodes.

d) We show that disc graphs have neighborhood independence 5. Consider a node $v$. We need to show that in the unit circle around $v$ (i.e., $v$'s neighborhood), there can fit at most five nodes with pairwise distance $> 1$. Assume we try to place six nodes with pairwise distance $> 1$. Given an arbitrary point $u$, we partition the circle into six parts of equal size (as in the picture) such that $u$ is on one of the lines.



Two points in the same part (including the rim) have distance at most 1. So any node with distance $> 1$ from $u$ must be outside the two parts in which $u$ is located. But then there are only four parts left where one can place the remaining five points, so two of them have to be in the same part and therefore have distance at most 1.