Chapter 12
Massively Parallel Computations

Distributed Systems

Summer Term 2020

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Massively Parallel Computations

Challenges

- Moore's law does not hold for ever
- We can only increase computational power by increasing the parallelism
- We need algorithmic techniques to deal with immense amounts of data

Massively Parallel Graph Computations

- Many important applications require solving standard graph problems in very large graphs (e.g., search engines, shortest path computations, etc.)
- We need ways to perform graph computations in highly parallel settings:
  - Graph data is shared among many servers / machines
  - Each machine can only store a small part of the graph
  - Need techniques to split and parallelize computations among machines
  - Use communication to coordinate between the machines
- Related to (standard) distributed graph computations

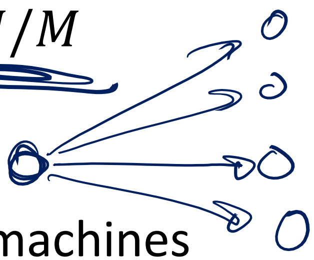
Massively Parallel Computation (MPC) Model

MPC Model

- An abstract formal model to study large-scale parallel computations
  - Aims to study parallelism at a more coarse-grained level than classic fine-grained parallel models like PRAM
    (models settings where communication is much more expensive than computation)

Formal Model

- Input of size N words (1 word $= O(\log N)$ bits, for graphs, $N = O(|E|)$)
- There are $M \ll N$ machines
- Each machine has a memory of S words, i.e., we need $S \geq N/M$
  - We typically assume that $S = N^c$ for a constant $c < 1$
- Time progresses in synchronous rounds, in each round,
  every machine can send & receive S words to & from other machines



- Initially, the data is partitioned in an arbitrary way among the M machines
  - Such that every machine has a roughly equal part of the data
  - W.l.o.g., data is partitioned in a random way among the machines

MPC Model for Graph Computations

$\tilde{O}(\cdot)$: hides polylog. factors

Assumption: Input is a graph $G = (V, E)$

- Number of nodes $n = |V|$, number of edges $m = |E|$, nodes have IDs
- Input can be specified by the set E of edges
  - each edge might have some other information, e.g., a weight
  - for simplicity, assume that every node has degree ≥ 1
- Initially, each edge is given to a uniformly random machine
- We typically assume that $S = \tilde{O}(N/M) = \tilde{O}(m/M)$

$S \leq \frac{N}{M} \cdot \text{polylog}(n)$

Strongly superlinear memory regime

nodes

$$S = n^{1+\varepsilon} \text{ for a constant } \varepsilon > 0$$

Strongly sublinear memory regime

$$S = n^{\alpha} \text{ for a constant } 0 < \alpha < 1$$

Near-linear memory regime

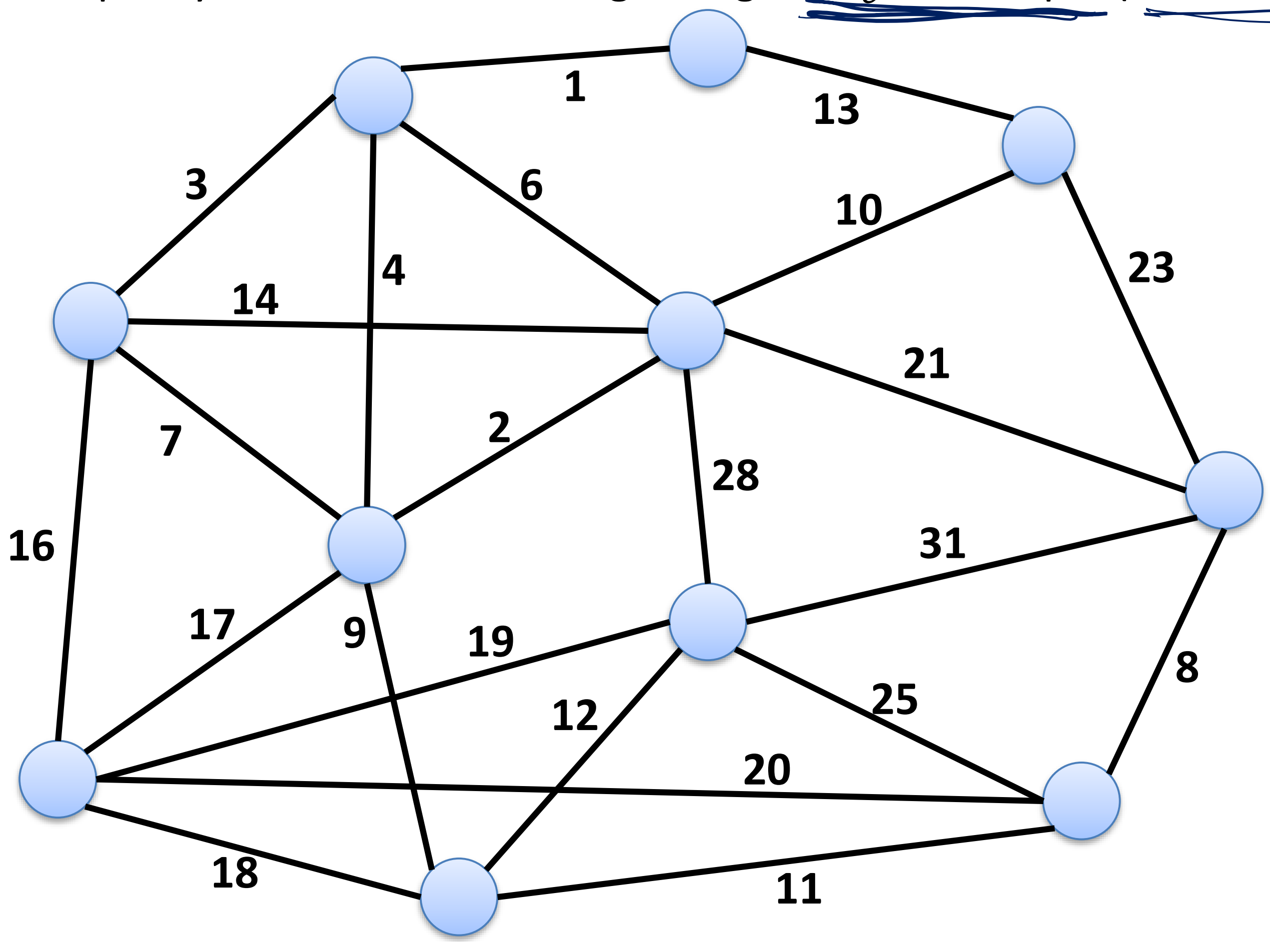
$$S = \tilde{O}(n)$$

Minimum Spanning Tree (MST) Problem

Given: connected graph $G = (V, E)$ with edge weights w_e

Goal: find a spanning tree $T = (V, E_T)$ of minimum total weight

- For simplicity, assume that the edge weights w_e are unique (makes MST unique)

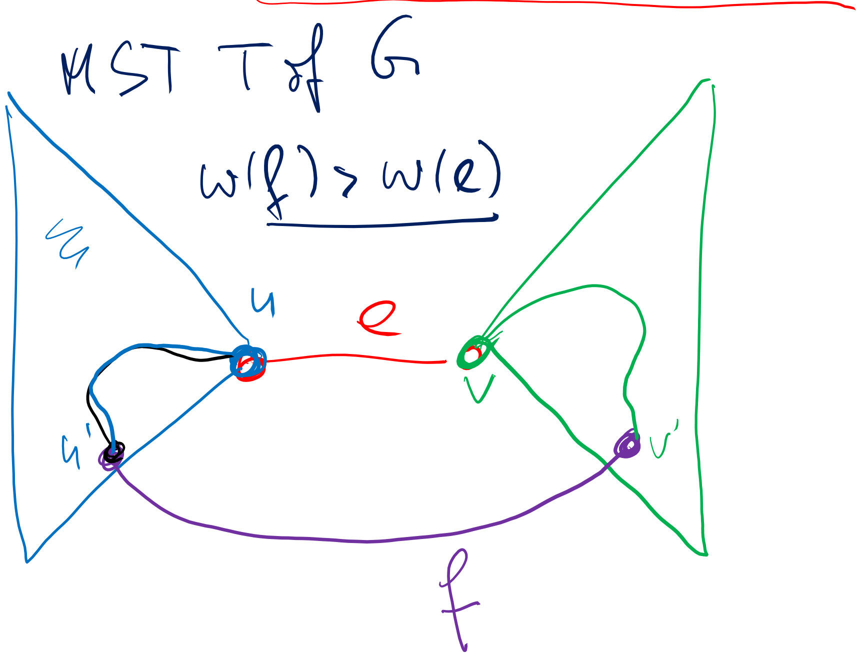


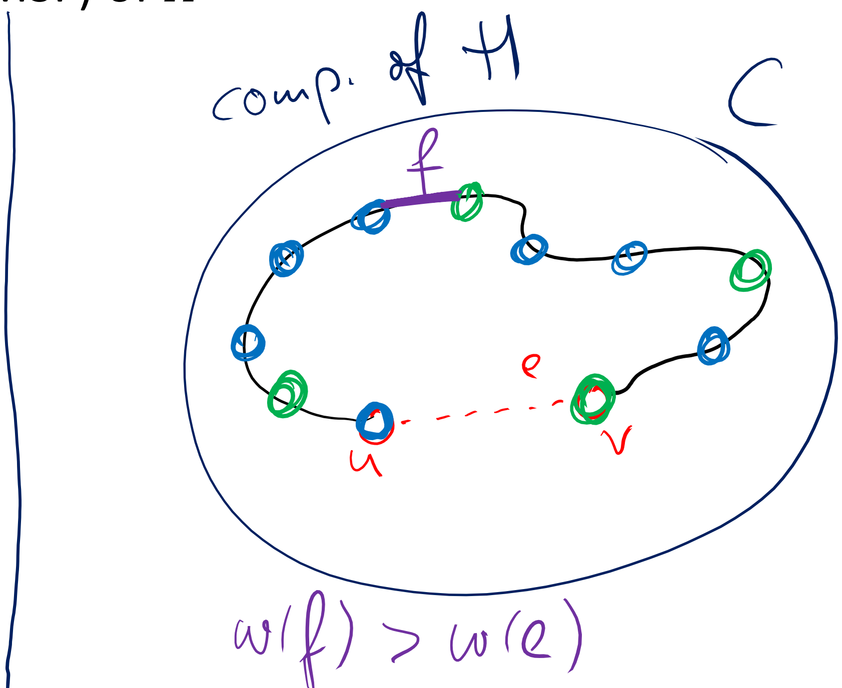
Properties of the MST

Minimum Spanning Forest (MSF) of G:

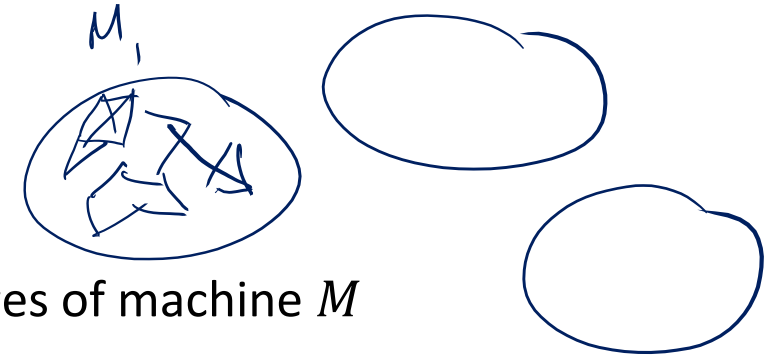
- A forest consisting of the MST of each of the connected components of G
  - Maximal forest of minimum total weight

Claim: Let $G = (V, E, w)$ be a weighted graph and let $H = (V', E', w)$ be a subgraph of G. If $e \in E'$ is an edge of the MST (or MSF) of G, then e is also an edge of the minimum spanning forest (MSF) of H





MST With $\underbrace{\text{Strongly Superlinear}}_{S=n^{1+\varepsilon}}$ Memory



Initially:

- Each machine has $O(n^{1+\varepsilon})$ edges
  - There are $M = O(m/n^{1+\varepsilon})$ machines
- Let H_M be the subgraph induced by the edges of machine M

MPC Algorithm:

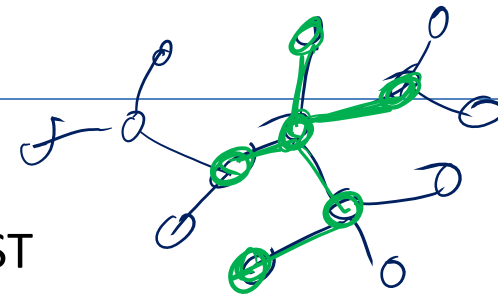
1. Each machine M computes minimum spanning forest F_M of H_M
2. Discard all edges that are not part of some MSF F_M
3. Remaining number of edges:
$$m' \leq M \cdot n = O(m/n^{\varepsilon})$$
4. Redistribute remaining edges to $M' = O(m'/n^{1+\varepsilon})$ machines
   - Randomly reassign each edge

$m \leq n^2, \; n^{2-\varepsilon}, \; n^{2-2\varepsilon}, \; \ldots$

- Algorithm reduces number of edges by factor $\Theta(n^{\varepsilon})$ in 1 round.
- $O(1/\varepsilon)$ repetitions suffice to solve the problem

$\Longrightarrow O(1)$ rounds

Borůvka's MST Algorithm



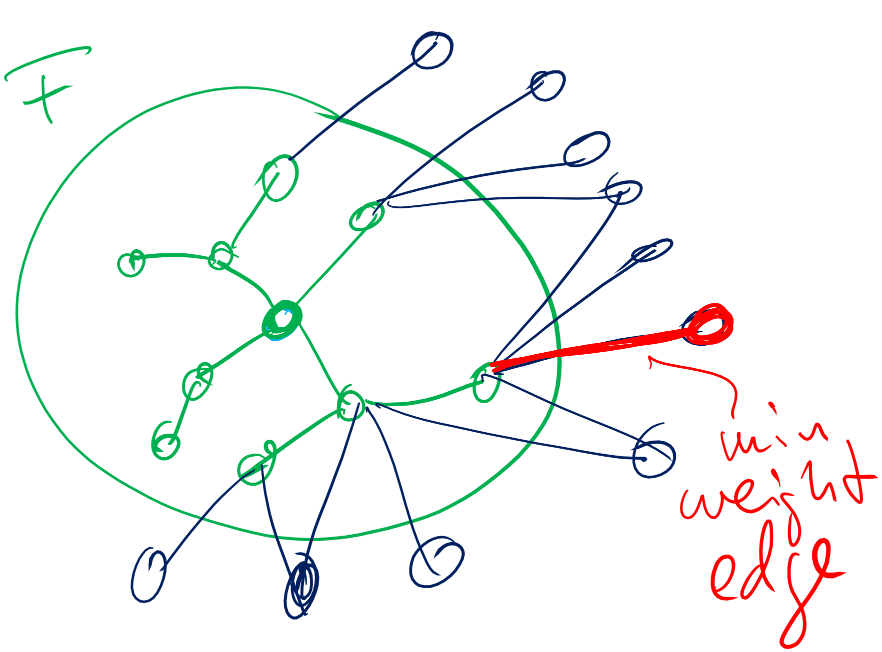
MST Fragment:

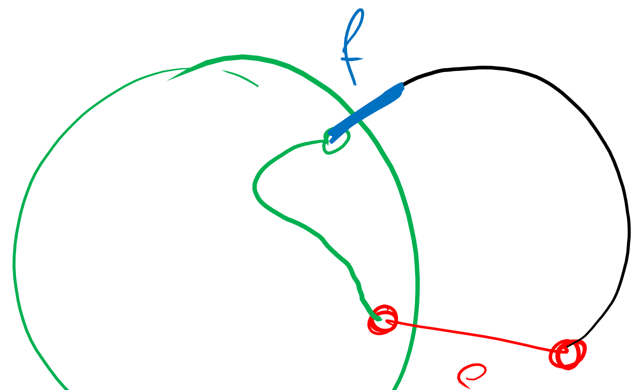
- A connected subtree $F = (V_F, E_F)$ of the MST

Minimum edge of MST fragment $\boldsymbol{F = (V_F, E_F)}$:

- Minimum weight edge connecting a node in V_F with a node in $V \setminus V_F$

Lemma: For every MST fragment F, the minimum edge of F is in the MST





$w(f) > w(e)$

Borůvka's MST Algorithm

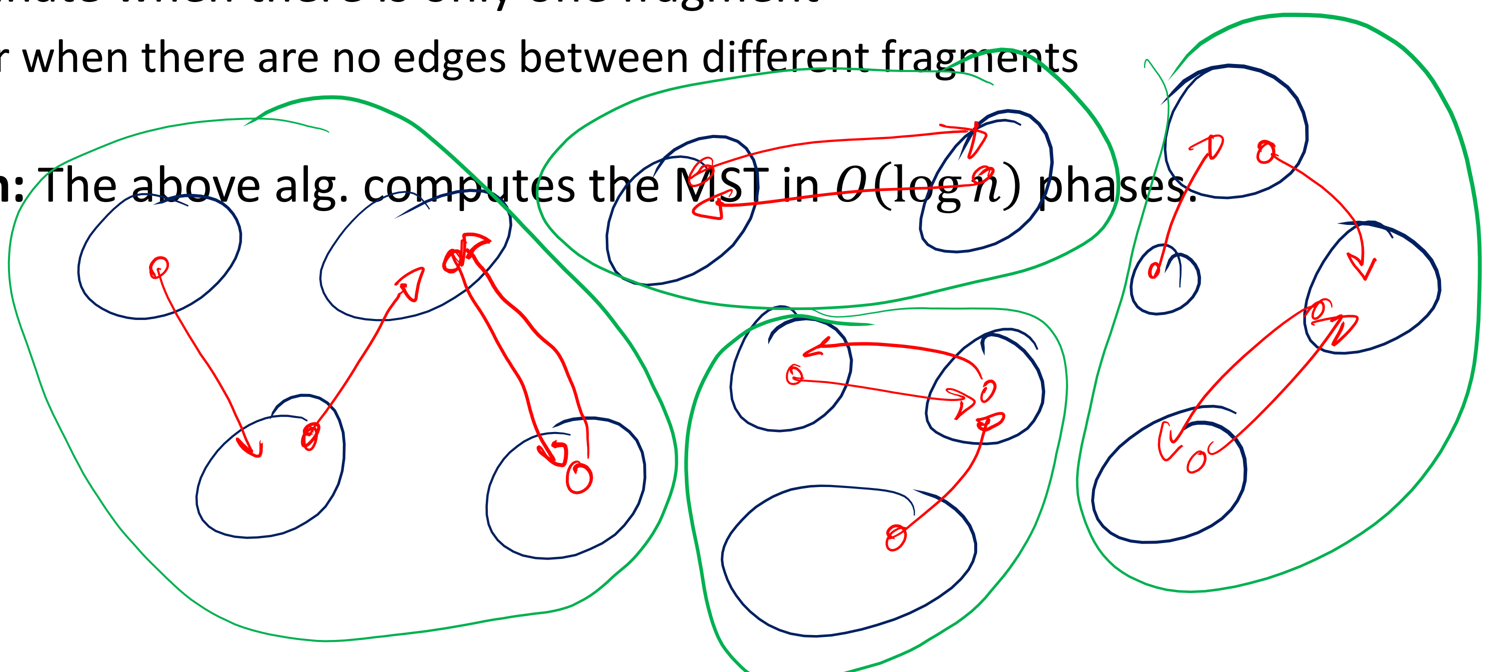
Algorithm description:

- Develops the MST in parallel phases
- Initially, each node is an MST fragment of size 1 (and with no edges)
- In each phase:

  add the minimum edge of each fragment to the MST

- Terminate when there is only one fragment
  - or when there are no edges between different fragments

Theorem: The above alg. computes the MST in $O(\log n)$ phases.



MST With Strongly Sublinear Memory: Ideas

Assume: $G = (V, E)$ with n nodes, m edges, memory $S = n^{\alpha}$ for const. $\alpha > 0$

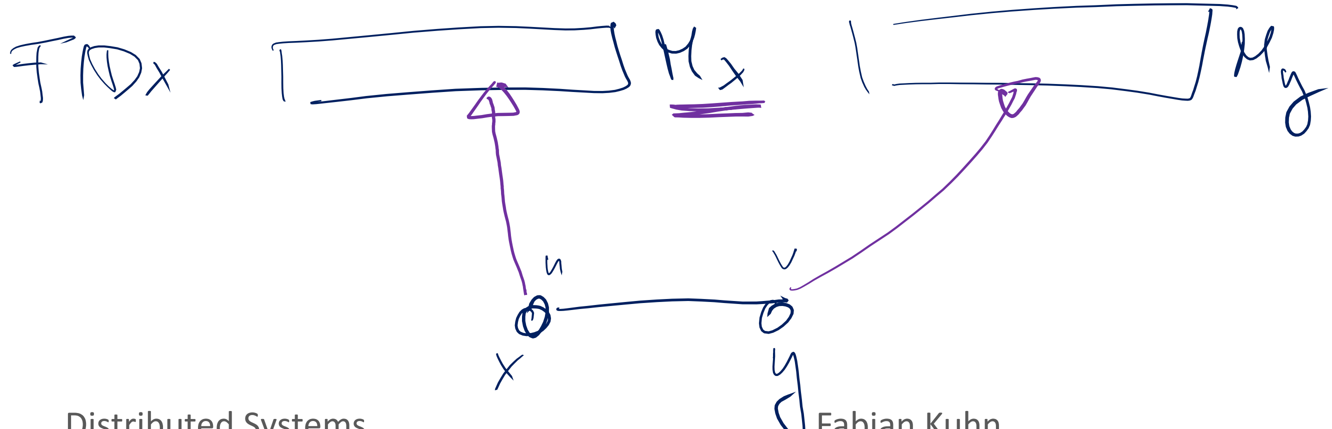
- Also assume that we have $M \geq m/S \cdot c \log n$ machines for suff. large $c \geq 1$

Representation of algorithm state:

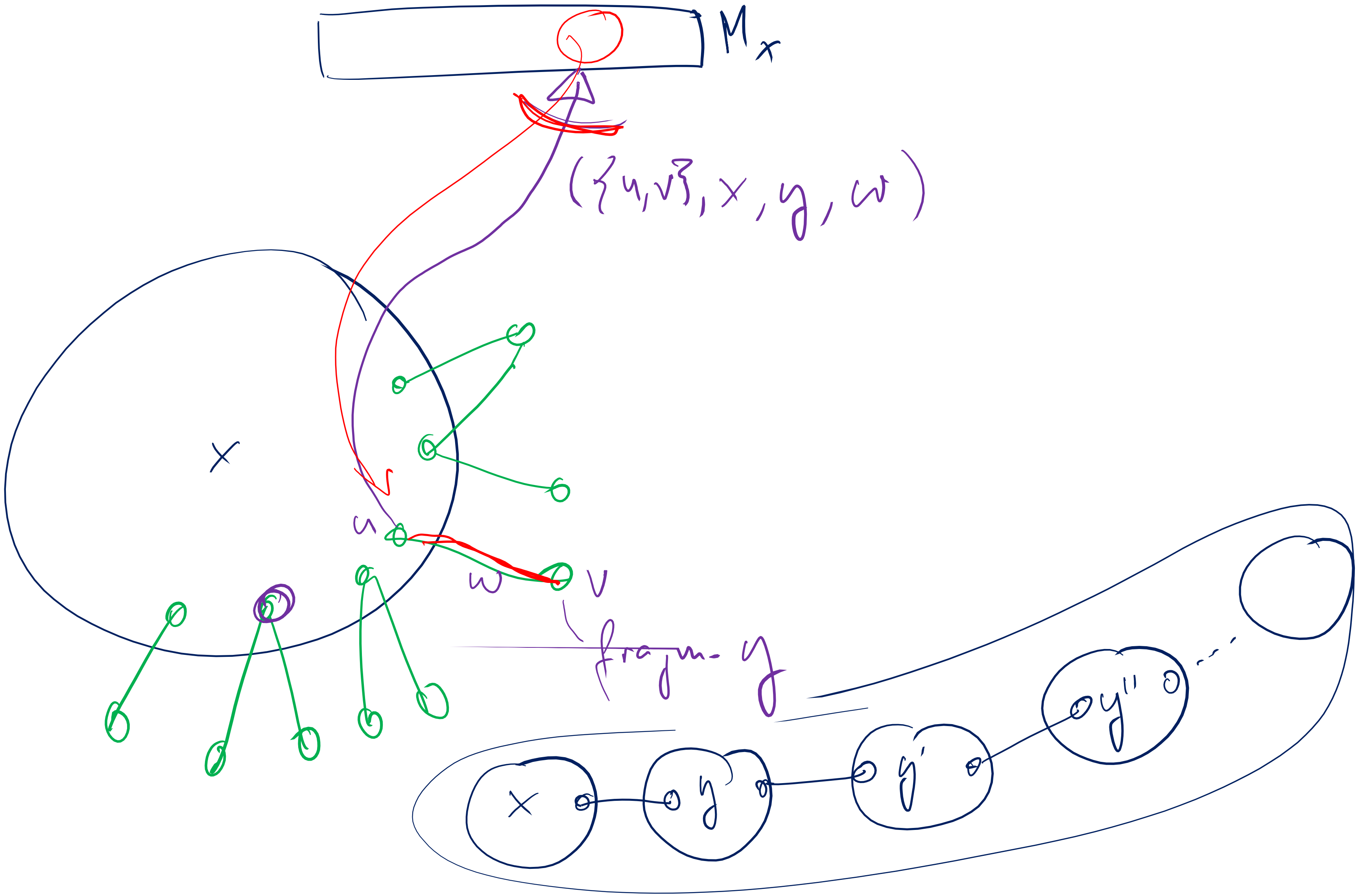
- Each fragment has a unique ID, fragment ID of node u: $\mathrm{FID}(u)$
- The machine storing an edge $\{u, v\}$ knows the fragment IDs of u and v

Goal: implement one phase in time $O(1)$:

- Assume that for each fragment ID x, there is some responsible machine M_x
  - Additional empty machines that are randomly assigned (e.g. by a hash function)
- For now, assume that each node u directly interacts with machine $M_{\mathrm{FID}(u)}$

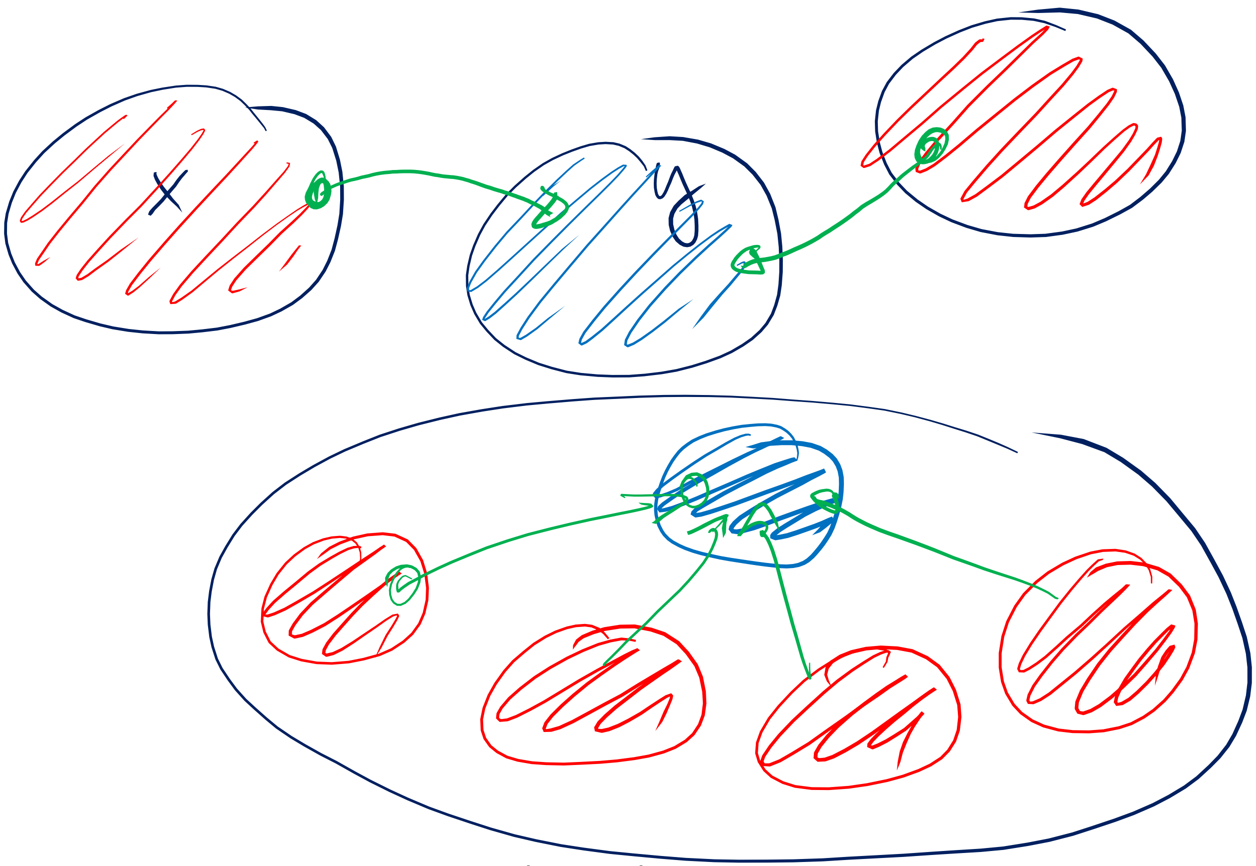


Implementing One Phase (First Attempt)



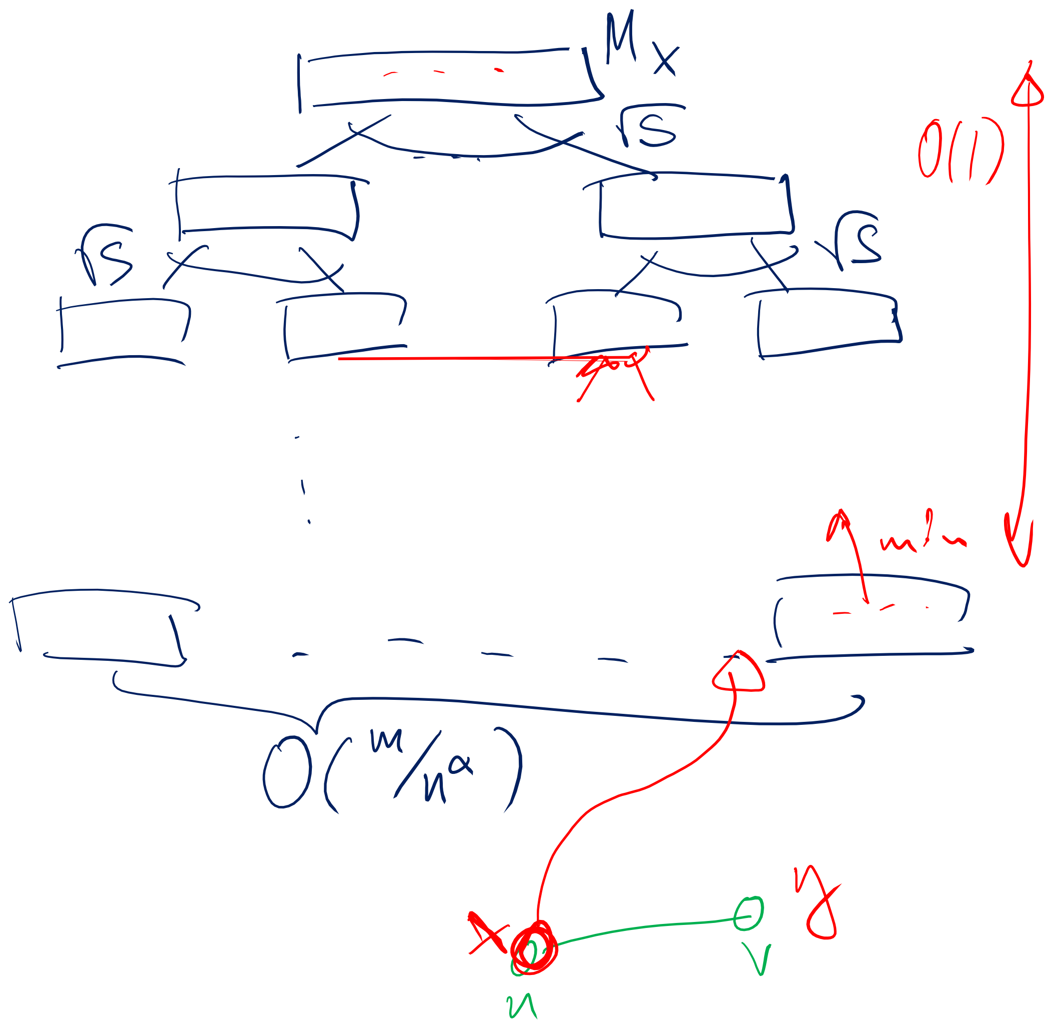
Small Change to the Basic Algorithm

- In each phase, each fragment initially picks a random color in $\{\text{red}, \text{blue}\}$
- Let $\{u, v\}$ be the minimum edge of a fragment F
- Only add $\{u, v\}$ to MST in current phase if F is a red fragment and $\{u, v\}$ connects to a blue fragment.



Implementation with Aggregation Trees

for each fragment x



MST with Strongly Sublinear Memory

Theorem: In the strongly sublinear memory regime (i.e., when $S = n^{\alpha}$ for a constant $\alpha \in (0,1)$), an MST can be computed in time $O(\log n)$.

MST in the Near-Linear Memory Regime

- Assume that $S = n \cdot (\log n)^c$ for a sufficiently large constant $c > 0$.
- Instead of MST, we consider a simpler, closely related probem

Connectivity / Component Identification

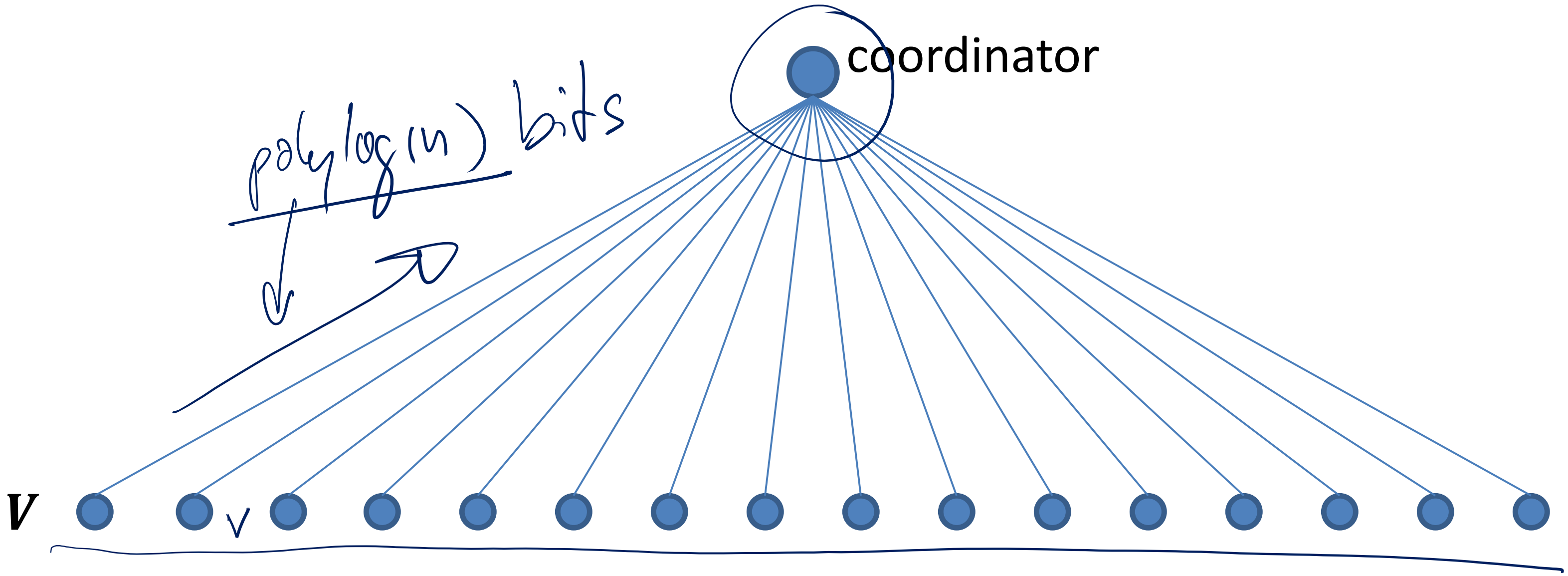
- At the end, algorithm needs to output a number $C(u)$ for each node $u \in V$ such that $C(u) = C(v)$ iff u and v are in the same connected component of G.

Observations

- Algorithm in particular allows to compute whether G is connected
- The MST algorithm from before can be used to solve component identification
  - The algorithm terminates when there are no more edges connecting different fragments. The fragment IDs at the end can be used as outputs
- In combination with some binary search over the edge weights, component identification can be used to also compute an MST
  - Everything we will do can be extended to the MST problem (at the cost of maybe a couple of log-factors in the required memory per machine)

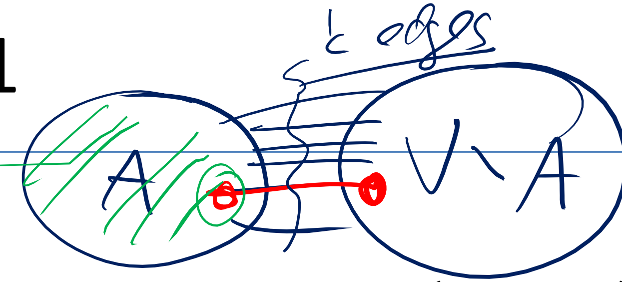
The Single-Round Coordinator Model

- We will study the problem in a different communication model



- There is a coordinator and one node for each $v \in V$
- Node v initially knows the set of its neighbors (i.e., all incident edges)
- Each node $v \in V$ is allowed to send one message to the coordinator
- Afterwards the coordinate needs to be able to compute the output
- We will assume that the nodes have access to shared randomness
- We will use the graph sketching technique

Graph Sketching: Warm Up 1



Single Cut Problem:

- Fix $A \subseteq V$. Assume that there are $k \geq 1$ edges across the cut $(A, V \setminus A)$.
- **Goal:** Coordinator needs to return one of the k edges across the cut

Assume first that $k = 1$:

- Define a unique ID for each edge $e = \{u, v\} \in E$: $\mathrm{ID}(e) = \mathrm{ID}(u) \circ \mathrm{ID}(v)$
- Each node $u \in A$ computes XOR_u as

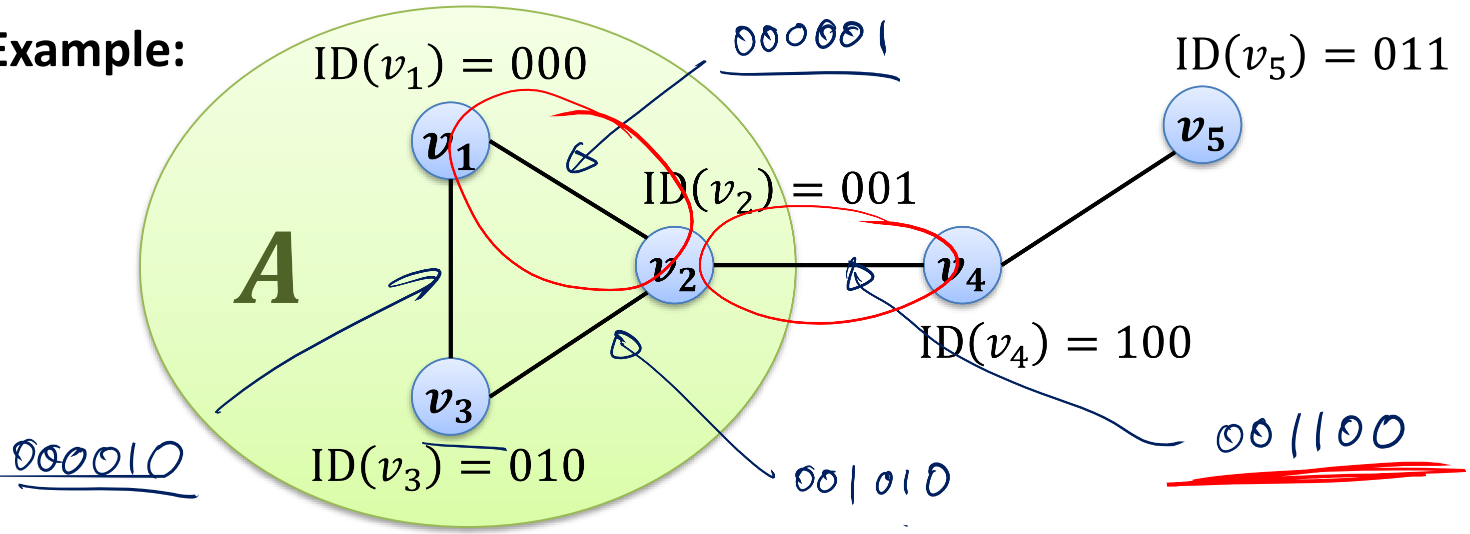
$$\mathrm{XOR}_u := \bigoplus_{e \in E : u \in e} \mathrm{ID}(e)$$

- Each node $u \in V$ sends XOR_u to coordinator
- Coordinator computes

$$\mathrm{XOR}_A := \bigoplus_{u \in A} \mathrm{XOR}_u$$

Graph Sketching: Warm Up 1

Example:



$$XOR_{v_1} = 000001 \oplus 000010 = 000011$$

$$XOR_{v_2} = 000001 \oplus 001010 \oplus 001100 = 000111$$

$$XOR_{v_3} = 000010 \oplus 001010 = 001000$$

$$XOR_{A} = 000011 \oplus 000111 \oplus 001000 = 001100$$

Graph Sketching: Warm Up 2

Assume that k is an arbitrary value

- Let E_A be the set of edges across the cut $(A, V \setminus A)$ $(|E_A| = k)$

Claim: If we use the same algorithm, $\mathrm{XOR}_A = \bigoplus_{e \in E_A} \mathrm{ID}(e)$.

Assume that we are given an estimate $\hat{k}$ s.t. $\frac{\hat{k}}{2} \leq k \leq \hat{k}$:

- Sample each edge with probability $1/\hat{k}$ and apply alg. with sampled edges

exp. # sampled edges in E_A is in $[1/2, 1]$

Graph Sketching: Warm Up 2

Assume that $\boldsymbol{k > 1}$ and an estimate $\boldsymbol{\hat{k}}$ s.t. $\boldsymbol{\frac{\hat{k}}{2} \leq k \leq \hat{k}}$ is given

- Sample each edge with probability $1/\hat{k}$
- Let E'_A be the sampled edges of E_A (across the cut)

Claim: $\mathbb{P}(|E'_A| = 1) \geq 1/10.$

$$\mathbb{P}(|E'_A| = 1) = k \cdot \frac{1}{\hat{k}} \cdot \left(1 - \frac{1}{\hat{k}}\right)^{k-1}$$

$$\geq \frac{\hat{k}}{2} \cdot \frac{1}{\hat{k}} \cdot \left(1 - \frac{1}{\hat{k}}\right)^{\hat{k}}$$

$$\geq \frac{1}{2} \cdot 4^{-\frac{1}{\hat{k}} \cdot \hat{k}}$$

$$\geq \frac{1}{10}.$$

Graph Sketching: Warm Up 2

Discussion:

- How can we sample each edge with probability $1/\hat{k}$?
  - Use shared randomness
- If we use the same algorithm, XOR_A is equal to an edge of E_A if $|E_A'| = 1$

How can we distinguish $\boldsymbol{|E_A'| = 1}$ from $\boldsymbol{|E_A'| \neq 1}$?

- We need to make sure that
  a) The bit-wise XOR of 0 or > 1 edge IDs is not equal to an edge ID
  b) Edge IDs can be distinguished from the XORs of 0 or > 1 edge IDs

Random Edge IDs

Edge ID of edge $\boldsymbol{e = \{u, v\} \in E}$ (assume $\mathbf{ID}(\boldsymbol{u}) < \mathbf{ID}(\boldsymbol{v})$)

$$\mathbf{ID}(\boldsymbol{e}) = \mathbf{ID}(\boldsymbol{u}) \circ \mathbf{ID}(\boldsymbol{v}) \circ \boldsymbol{R_e}$$

- R_e is a random bit string of length $80 \ln n$ where each bit is 1 with prob. $1/8$
- Let R'_A be the bitwise XOR of R_e for $e \in E'_A$

Claim: Let X be the number of 1s in R'_A. If $|E'_A| = 0$, then $X = 0$, otherwise
- If $|E'_A| = 1$, then $1 < X < 14 \ln n$ with high probability
- If $|E'_A| > 1$, then $X > 14 \ln n$ with high probability

Proof Sketch:

Random Edge IDs

Claim: Let X be the number of 1s in R'_A. If $|E'_A| = 0$, then $X = 0$, otherwise

- If $|E'_A| = 1$, then $1 < X < 14 \ln n$ with high probability
- If $|E'_A| > 1$, then $X > 14 \ln n$ with high probability

Proof Sketch:

- If $|E'_A| \geq 2$, each of the $80 \ln n$ bits of R'_A is 1 with prob. $\geq 2 \cdot \frac{1}{8} \cdot \frac{7}{8} > \frac{1}{5}$

Connected Components with Graph Sketching

One phase of the Borůvka algorithm

- We need to find one outgoing edge for each fragment
  - Then the coordinator can add a subset of these edges and reduce the number of fragments by a factor 2
- We do not know the number of out-going edges of the different fragments
  - And different fragments might have different numbers
- Use different sampling probabilities: $\frac{1}{n}, \frac{2}{n}, \frac{4}{n}, \dots, \frac{1}{2}$ and send sketches for all probabilities to coordinator
  - For each instance, each $v \in V$ sends XOR of sampled edges to coordinator
- For each fragment, one of the probabilities succeeds with probability $\geq 1/10$
- When having $\Theta(\log n)$ instances for each of the probabilities, we get an outgoing edge for each fragment with high probability
- Each node can send $O(\log^3 n)$ bits to coordinator for one phase

Observation: The protocol does not depend on the fragments

- We can therefore send the information for all phases in parallel

Connected Components with Graph Sketching

Theorem: In the coordinator model, there is a protocol where every node $v \in V$ send $O(\log^4 n)$ bits to the coordinator s.t. the coordinator can solve the connectivity & connected components problem.

Remarks:

- The number of bits can be reduced to $O(\log^3 n)$
  - It is sufficient to succeed with constant prob. for each fragment in each phase
- $\Omega(\log^3 n)$ bits are necessary [Nelson, Yu; 2019]
- Graph sketching has been introduced by [Ahn, Guha, McGregor; 2012]

Implementation in the MPC Model

1. For every node $v \in V$, create a responsible machine M_v
   - Send each edge $\{u, v\}$ to both M_u and M_v
   - Make sure that each machine gets $\tilde{O}(n)$ edges

1. The randomness for each edge can be generated initially by the machine that holds the edge
   - Also send the randomness for the edge $\{u, v\}$ to M_u and M_v

2. Use one additional machine for the coordinator

Theorem: In the MPC model with $S = \tilde{O}(n)$, the connectivity & connected components problem can be solve in $O(1)$ rounds.

Discussion

- Graph sketching can help in many different contexts, e.g.,
  - also in the strongly-sublinear memory regime to save communication
  - in the streaming model
  - in the standard distributed model to save message

- In the strongly sublinear memory regime, it is not known whether it is possible to be faster than $O(\log n)$ rounds
  - It is widely believed that there should be an $\Omega(\log n)$ lower bound
  - Even the following simple version of the problem seems to require $\Omega(\log n)$ time

    distinguish 2 cycles of length $n/2$ from one cycle of length n