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Theoretical Computer Science - Bridging Course
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Revision Sheet

Exercise 1: The class $\mathcal{NPC}$

Given a set U of n elements ('universe') and a collection $S \subseteq \mathcal{P}(U)$ of subsets of U, a selection $C_1, \dots, C_k \in S$ of k sets is called a *set cover* of (U, S) of size k if $C_1 \cup \ldots \cup C_k = U$.
Show that the problem

$$\textsc{SetCover} := \{\langle U, S, k\rangle \,|\, U \text{ is a set, } S \subseteq \mathcal{P}(U) \text{ and there is a set cover of } (U,S) \text{ of size } k\}$$

is NP-complete.
You may use that

$$\textsc{DominatingSet} = \{\langle G, k\rangle \mid G \text{ has a dominating set with } k \text{ nodes}\}.$$

is NP-complete. A subset of the nodes of a graph G is a *dominating set* if every other node of G is adjacent to some node in the subset.

Exercise 2: Complexity Classes: Big Picture

(a) Why is $\mathcal{P} \subseteq \mathcal{NP}$?

(b) Show that $\mathcal{P} \cap \mathcal{NPC} = \emptyset$ if $\mathcal{P} \neq \mathcal{NP}$.
Hint: Assume that there exists a $L \in \mathcal{P} \cap \mathcal{NPC}$ and derive a contradiction to $\mathcal{P} \neq \mathcal{NP}$.

(c) Give a Venn Diagram showing the sets $\mathcal{P}, \mathcal{NP}, \mathcal{NPC}$ for both cases $\mathcal{P} \neq \mathcal{NP}$ and $\mathcal{P} = \mathcal{NP}$.
Remark: Use the results of (a) and (b) even if you did not succeed in proving those.

Exercise 3: CFL, PDA and Non-Context Free Languages

Using Pumping Lemma, show that the following languages are not CFL.

(a) $L = \{a^i b^j c^k \mid i < j \text{ and } i < k\}$

(b) Knowing that $L' = \{a^i b^j c^k \mid i < j\}$ is a CFL, are context free languages closed under intersection?

(c) Create a pushdown automaton that accepts the language $\{0^{2n}1^n \mid n > 0\}$.

Exercise 4: Decidability

1. Show that $A = \{\langle R, S\rangle \mid R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S)\}$ is decidable.
2. Show that $EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are Turing Machines and } L(M_1) = L(M_2)\}$ is undecidable.

   Hint: You may use that $E_{TM} = \{\langle M\rangle \mid M \text{ is a Turing Machine and } L(M) = \emptyset\}$ is undecidable.

Exercise 5: Graphs

A graph $G = (V, E)$ is said to be connected if for every pair of nodes $u, v \in V$ such that $u \neq v$ there exists a path in G connecting u to v. A tree is a simple, connected graph without cycles.

1. Prove that if G is connected, then for any two non empty subsets V_1 and V_2 of V such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, there exists an edge joining a vertex in V_1 to a vertex in V_2.
2. Show that a tree with n nodes has $n - 1$ edges.
   Hint: You may use that a tree has at least one leaf, i.e., a node of degree one.