

Theoretical Computer Science - Bridging Course

Summer Term 2020

Exercise Sheet 9

for getting feedback submit electronically by 12:15, Tuesday, July 14, 2020

Exercise 1: Propositional Logic: Basic Terms (2+2+2+2 Points)

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \rightarrow \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I . In case $I \models \varphi$, I is called a *model* for φ .

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \wedge \neg q) \vee (\neg p \vee q)$

(b) $\varphi_2 = (\neg p \wedge (\neg p \vee q)) \leftrightarrow (p \vee \neg q)$

(c) $\varphi_3 = (p \wedge \neg q) \rightarrow \neg(p \wedge q)$

(d) $\varphi_4 = (p \wedge q) \rightarrow (p \vee r)$

Remark: $a \rightarrow b := \neg a \vee b$, $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$, $a \not\rightarrow b := \neg(a \rightarrow b)$.

Exercise 2: CNF and DNF (2+2 Points)

(a) Convert $\varphi_1 := (p \rightarrow q) \rightarrow (\neg r \wedge q)$ into Conjunctive Normal Form (CNF).

(b) Convert $\varphi_2 := \neg((\neg p \rightarrow \neg q) \wedge \neg r)$ into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Exercise 3: Logical Entailment (2+2 Points)

A *knowledge base* KB is a set of formulae over a given set of atoms Σ . An interpretation I of Σ is called a *model* of KB , if it is a model for *all* formulae in KB . A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ .

Let $KB := \{p \vee q, \neg r \vee p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b) $\varphi_2 := (q \leftrightarrow r) \rightarrow p$

Exercise 4: Inference Rules and Calculi

(2+2 Points)

Let $\varphi_1, \dots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \dots, \varphi_n}{\psi}$$

means that if $\varphi_1, \dots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* \mathbf{C} is described by a *set* of inference rules.

Given a formula ψ and knowledge base $KB := \{\varphi_1, \dots, \varphi_n\}$ (where $\varphi_1, \dots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus \mathbf{C} to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C}_1 : \frac{\varphi \rightarrow \psi, \psi \rightarrow \chi}{\varphi \rightarrow \chi}, \frac{\neg \varphi \rightarrow \psi}{\neg \psi \rightarrow \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \rightarrow \psi, \psi \rightarrow \varphi}$$
$$\mathbf{C}_2 : \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{(\varphi \wedge \psi) \rightarrow \chi}{\varphi \rightarrow (\psi \rightarrow \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

- (a) $\{p \leftrightarrow \neg r, \neg q \rightarrow r\} \vdash_{\mathbf{C}_1} p \rightarrow q$
(b) $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow s\} \vdash_{\mathbf{C}_2} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.