Exercise 1: Minimum Spanning Trees

Let $G = (V, E, w)$ be an undirected, connected, weighted graph with pairwise distinct edge weights.

(a) Show that $G$ has a unique minimum spanning tree.

(b) Show that the minimum spanning tree $T'$ of $G$ is obtained by the following construction:

\[ \text{Start with } T' = \emptyset. \text{ For each cut in } G, \text{ add the lightest cut edge to } T'. \]

Sample Solution

(a) Let $T$ and $T'$ be two minimum spanning trees with edges $e_1, \ldots, e_{n-1}$ and $e'_1, \ldots, e'_{n-1}$, sorted by increasing weight. Assume we have $T \neq T'$. Let $j$ be the largest index for which $e_j \neq e'_j$. As the weights are pairwise distinct, we also have $w(e_j) \neq w(e'_j)$. W.l.o.g. let $w(e_j) < w(e'_j)$. The graph $T' \setminus \{e'_j\}$ has two connected components with nodes $S$ and $V \setminus S$. Let $e_k$ be the edge in $T$ connecting $S$ and $V \setminus S$. As $T'$ contains only one edge between $S$ and $V \setminus S$, it must hold $k \geq j$ (as $e_k = e'_k$ for $k > j$). As $(T' \setminus \{e'_j\}) \cup \{e_k\}$ is a spanning tree and $w(e'_j) > w(e_j) \geq w(e_k)$, it has a smaller weight than $T'$, contradicting that $T'$ is minimal.

(b) Let $T$ be the MST of $G$ and $T'$ the set containing the lightest cut edges.

$T' \subseteq T$: Let $s \in T'$, i.e., $s$ is the lightest cut edge of a cut $(S, V \setminus S)$ in $G$. Let $e$ be the edge of $T$ connecting $S$ and $V \setminus S$. If $e \neq s$, then $w(s) < w(e)$ and the spanning tree $(T \setminus \{e\}) \cup \{s\}$ would have a smaller weight than $T$, contradicting that $T$ is an MST. Hence we have $e = s$ and thus $s \in T$.

$T \subseteq T'$: Let $e \in T$. The graph $T \setminus \{e\}$ has two connected components which define a cut in $G$. With an exchange argument as above one can show that $e$ is the (unique) lightest cut edge of this cut, i.e., we have $e \in T'$.

Exercise 2: Travelling Salesperson Problem

Let $p_1, \ldots, p_n \in \mathbb{R}^2$ be points in the euclidean plane. Point $p_i$ represents the position of city $i$. The distance between cities $i$ and $j$ is defined as the euclidean distance between the points $p_i$ and $p_j$. A tour is a sequence of cities $(i_1, \ldots, i_n)$ such that each city is visited exactly once (formally, it is a permutation of $\{1, \ldots, n\}$). The task is to find a tour that minimizes the travelled distance. This problem is probably costly to solve.\footnote{The Travelling Salesperson Problem is in the class of \textit{NP}-complete problems for which it is assumed that no algorithm with polynomial runtime exists. However, this has not been proven yet.} We therefore aim for a tour that is at most twice as long as a minimal tour.
We can model this as a graph problem, using the graph $G = (V, E, w)$ with $V = \{p_1, \ldots, p_n\}$ and $w(p_i, p_j) := \|p_i - p_j\|_2$. Hence, $G$ is undirected and complete and fulfills the triangle inequality, i.e., for any nodes $x, y, z$ we have $w(\{x, z\}) \leq w(\{x, y\}) + w(\{y, z\})$. We aim for a tour $(i_1, \ldots, i_n)$ such that $w(p_{i_n}, p_{i_1}) + \sum_{j=1}^{n-1} w(p_{i_j}, p_{i_{j+1}})$ is small.

Let $G$ be a weighted, undirected, complete graph that fulfills the triangle inequality. Show that the sequence of nodes obtained by a pre-order traversal of a minimum spanning tree (starting at an arbitrary root) is a tour that is at most twice as long as a minimal tour.

**Sample Solution**

Let $R = (i_1, \ldots, i_n)$ be a minimal tour and $w(R) := w(p_{i_n}, p_{i_1}) + \sum_{j=1}^{n-1} w(p_{i_j}, p_{i_{j+1}})$. Let $T$ be an MST, $w(T) := \sum_{e \in T} w(e)$ its weight and $P_T$ its pre-order sequence of nodes. As the graph is complete, $P_T$ is also a tour.

We add points to $P_T$ as follows: If two subsequent nodes $u$ and $v$ are not connected in $T$ by a tree edge, we add between $u$ and $v$ all nodes on the shortest path from $u$ to $v$ in $T$ (these are all nodes from $u$ to the first common ancestor $w$ and from there to $v$). We write $P'_T$ for the sequence that we obtain (this is formally not a tour as points are visited more than once).

In $P'_T$, two subsequent nodes are neighbors in $T$, so we can consider this sequence as a sequence of edges in $T$. Each edge from $T$ is contained in $P'_T$ exactly twice (if you go from the last point back to the root). Thus we have $w(P'_T) = 2 \sum_{e \in T} w(e)$. The triangle inequality implies $w(P_T) \leq w(P'_T)$ and hence $w(P_T) \leq 2 \sum_{e \in T} w(e)$.

The minimal tour $R$ defines a spanning tree $T_R$ by taking the edges between subsequent nodes in $R$. As $T$ is the minimum spanning tree we have $w(T) \leq w(T_R) \leq w(T_R) + w(p_{i_n}, p_{i_1}) = w(R)$ and hence $w(P_T) \leq 2 \cdot w(R)$.

Remark: The above argumentation also works for the post-order traversal. However, if you want the tour to start at a predefined point, it is easiest to use this point as the root of a pre-order traversal.