Chapter 6
Consensus II

Theory of Distributed Systems

Fabian Kuhn

Overview

- Introduction
- Consensus #1: Shared Memory
- Consensus #2: Wait-free Shared Memory
- Consensus #3: Read-Modify-Write Shared Memory
- Consensus #4: Synchronous Systems
- Consensus #5: Byzantine Failures
- Consensus #6: A Simple Algorithm for Byzantine Agreement
- Consensus #7: The Queen Algorithm
- Consensus #8: The King Algorithm
- Consensus #9: Byzantine Agreement Using Authentication
- Consensus #10: A Randomized Algorithm
- Shared Coin

- Most slides by R. Wattenhofer (ETHZ)

Consensus More Formally

Setting:

- n processes/threads/nodes $v_1, v_2, \ldots, v_n$
- Each process has an input $x_1, x_2, \ldots, x_n \in \mathcal{D}$
- Each (non-failing) process computes an output $y_1, y_2, \ldots, y_n \in \mathcal{D}$

Agreement:

The outputs of all non-failing processes are equal.

Validity:

If all inputs are equal to x, all outputs are equal to x.

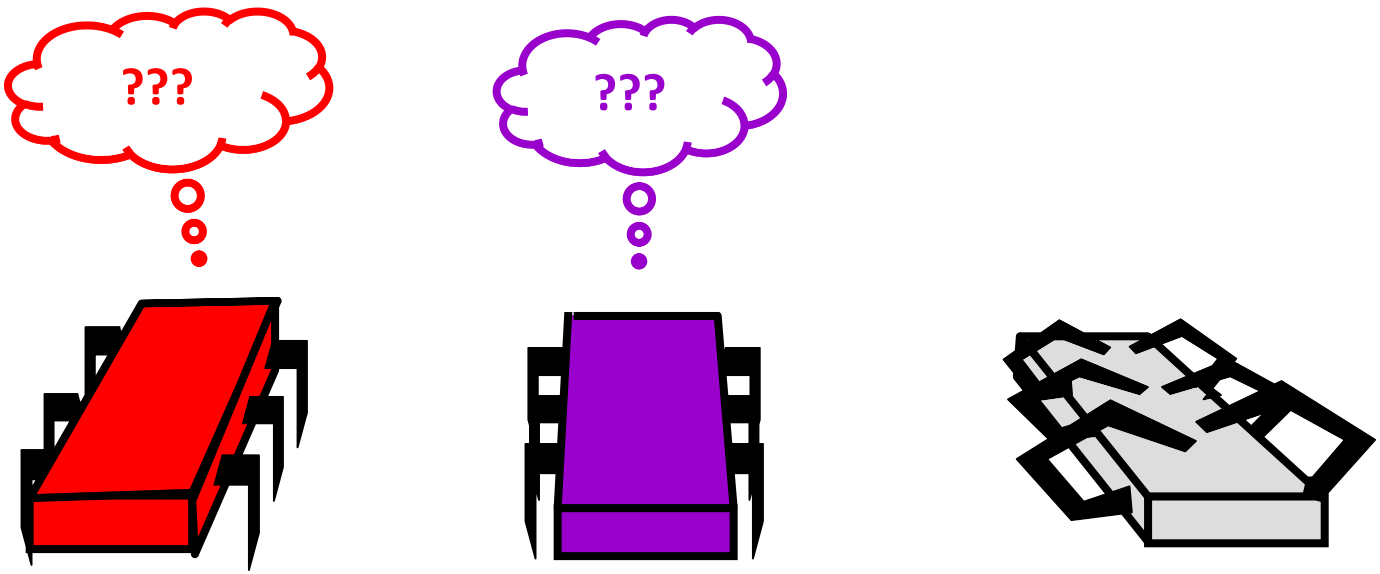
Termination:

All non-failing processes terminate after a finite number of steps.

Impossibility

Theorem

There is no deterministic asynchronous wait-free consensus algorithm using read/write atomic registers.

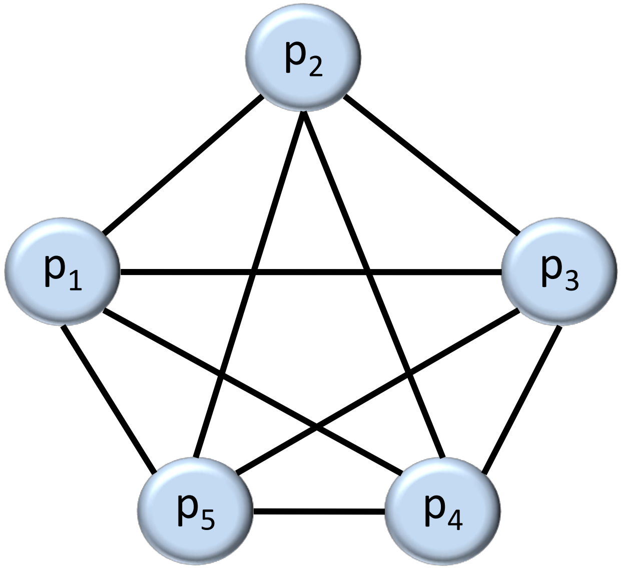


The Consensus Hierarchy

1	2	…	∞
• Read/Write Registers	• Test&Set • Fetch&Inc • Fetch&Add • Swap		• CAS • LL/SC

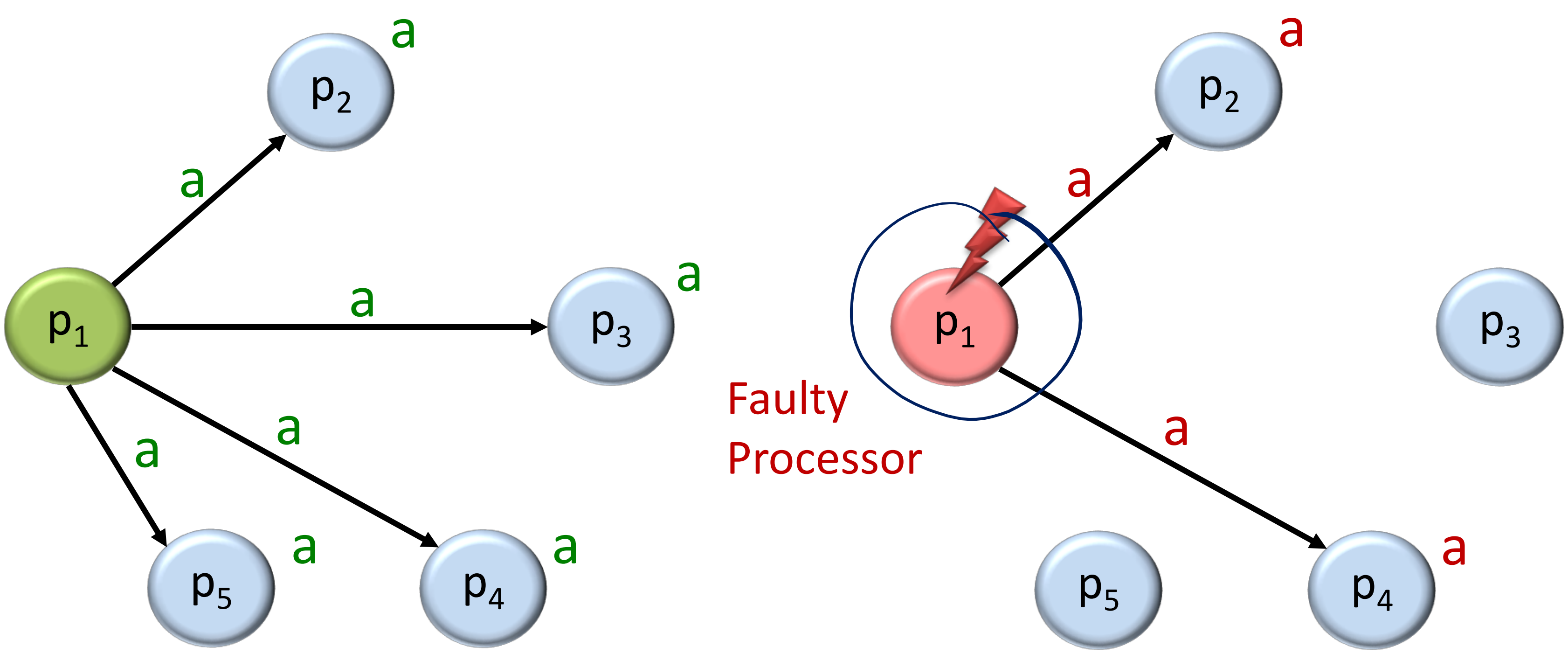
Consensus #4: Synchronous Systems

- One can sometimes tell if a processor has crashed
  - Timeouts
  - Broken TCP connections
- Can one solve consensus at least in synchronous systems?
- Model
  - All communication occurs in synchronous rounds
  - Complete communication graph

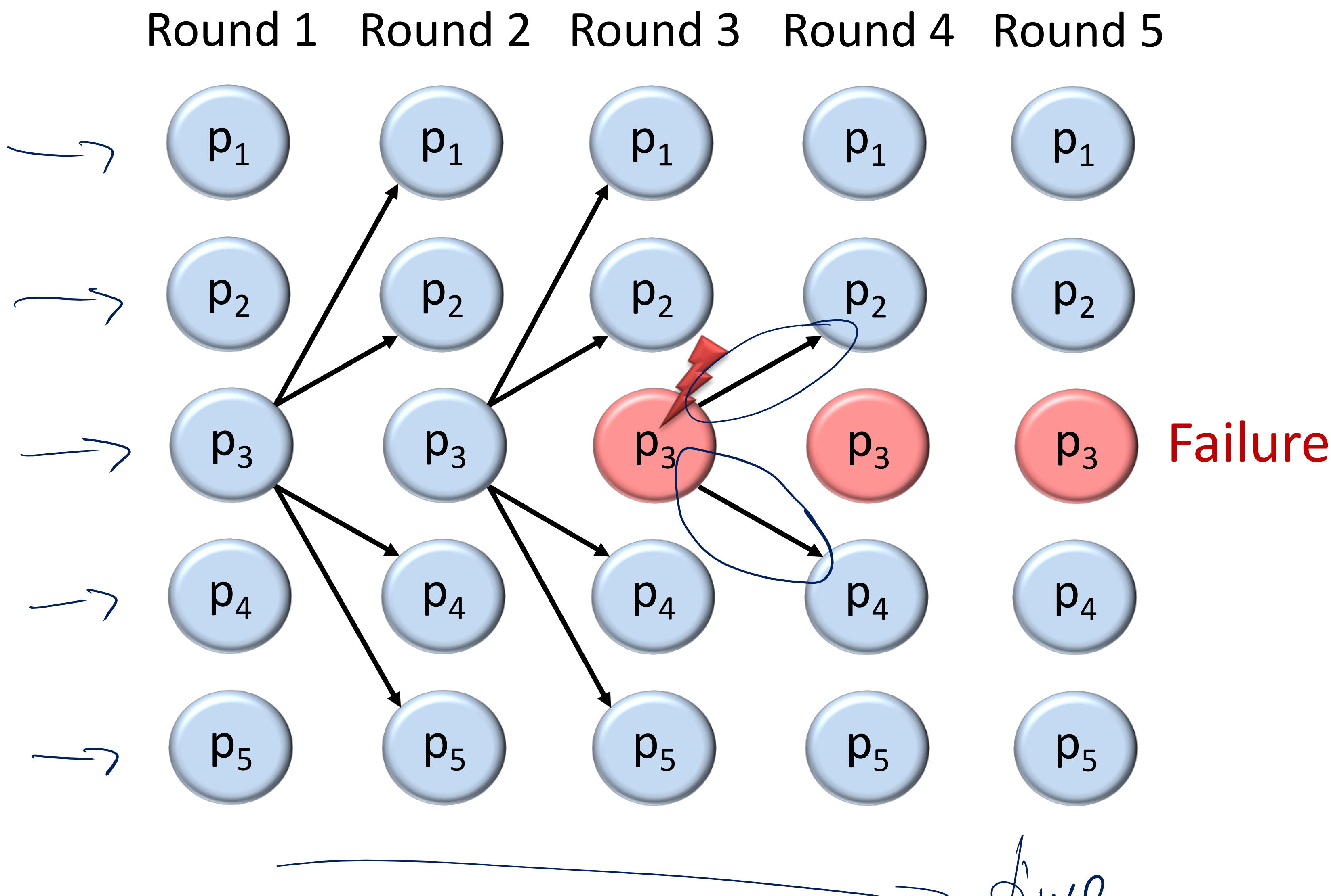


Crash Failures

- Broadcast: Send a message to all nodes in one round
  - At the end of the round everybody receives the message a
  - Every process can broadcast a value in each round
- Crash Failures: A broadcast can fail if a process crashes
  - Some of the messages may be lost, i.e., they are never received

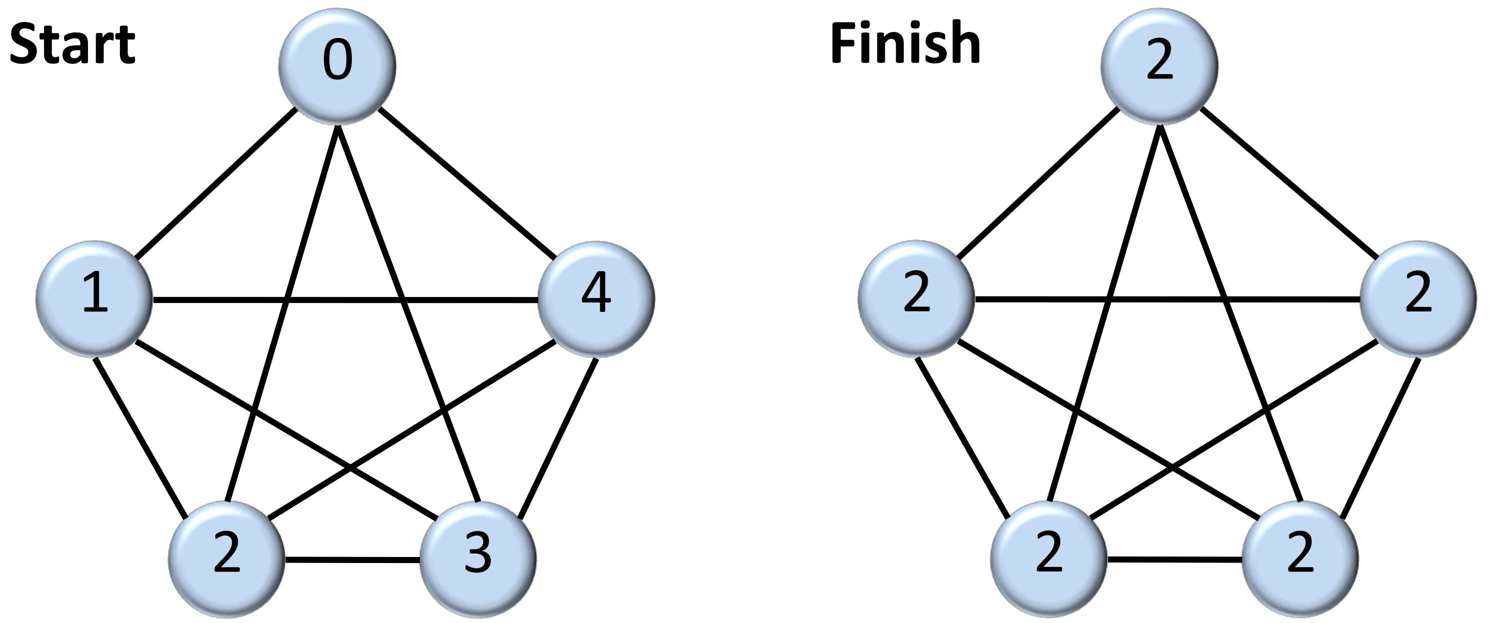


Process disappears after failure



Consensus Repetition

- **Input:** everybody has an initial value
- **Agreement:** everybody must decide on the same value



- **Validity conditon**: If everybody starts with the same value, everybody must decide on that value

A Simple Consensus Algorithm

Each process:

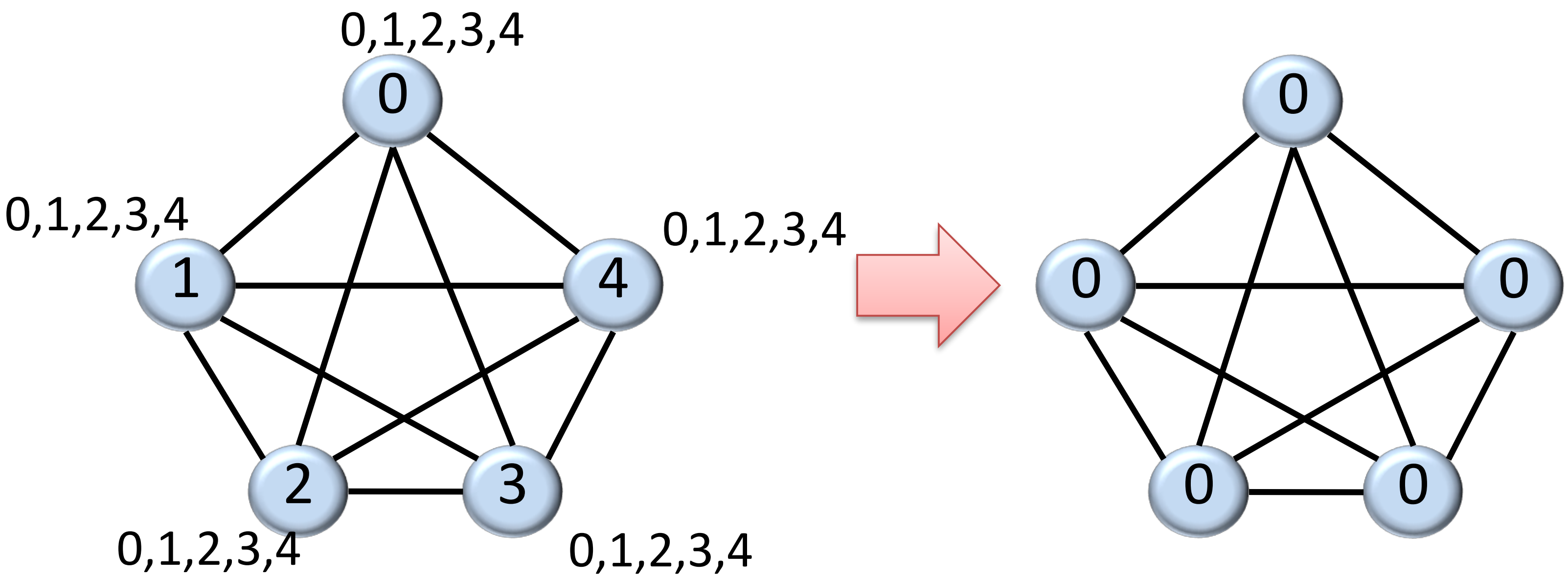
1. Broadcast own value
2. Decide on the minimum of all received values

Including the own value

Note than only one round is needed!

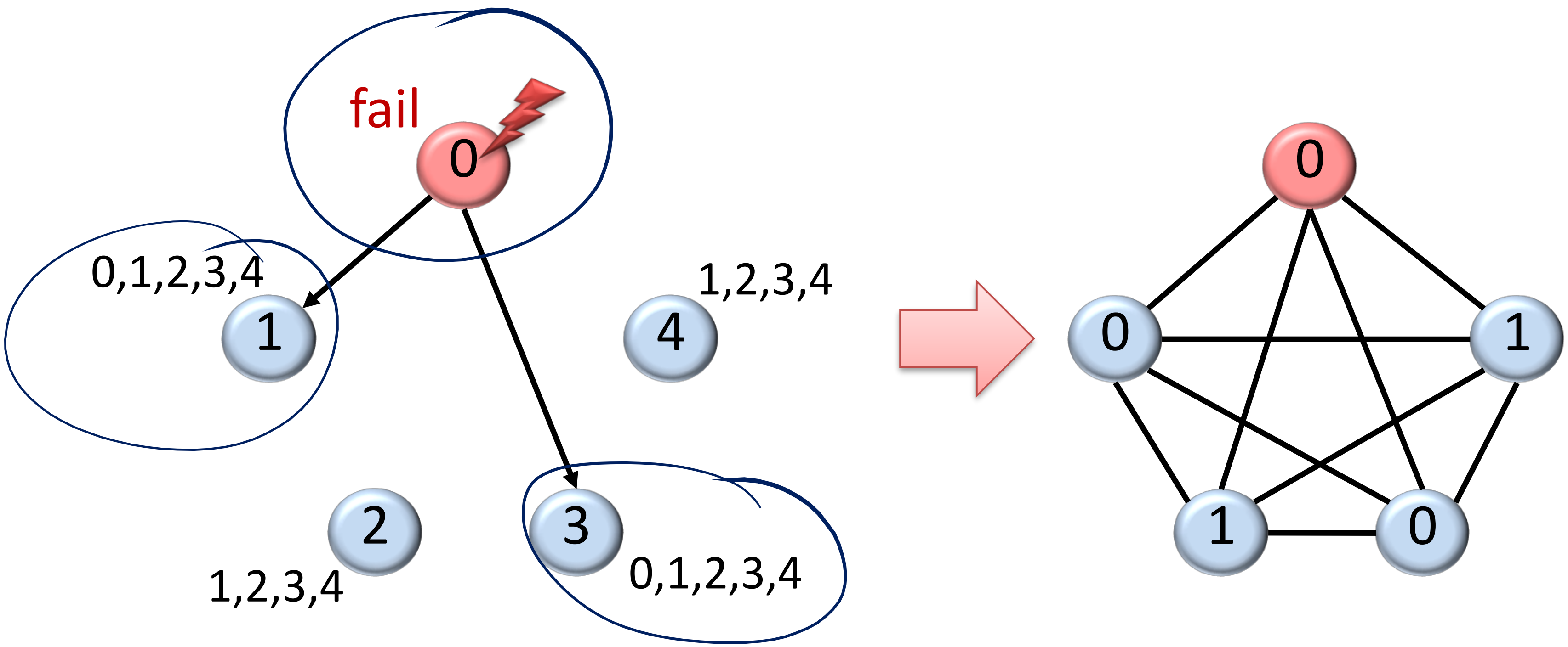
Execution Without Failures

- Broadcast values and decide on minimum → Consensus!
- Validity condition is satisfied: If everybody starts with the same initial value, everybody sticks to that value (minimum)



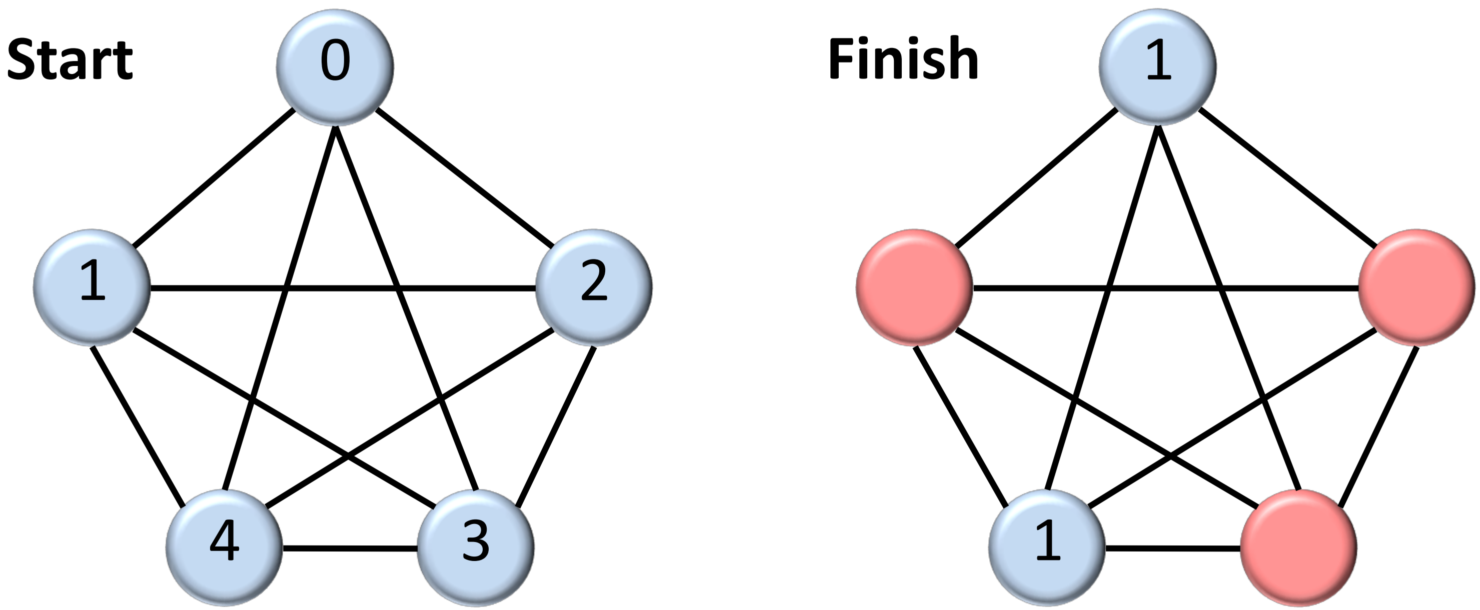
Execution With Failures

- The failed processor doesn't broadcast its value to all processors
- Decide on minimum → No consensus!



f-Resilient Consensus Algorithm

- If an algorithm solves consensus for f failed processes, we say it is an f-resilient consensus algorithm
- Example: The input and output of a 3-resilient consensus alg.



- **Refined validity condition**:
  All processes decide on a value that is available initially

An f-Resilient Consensus Algorithm

Each process:

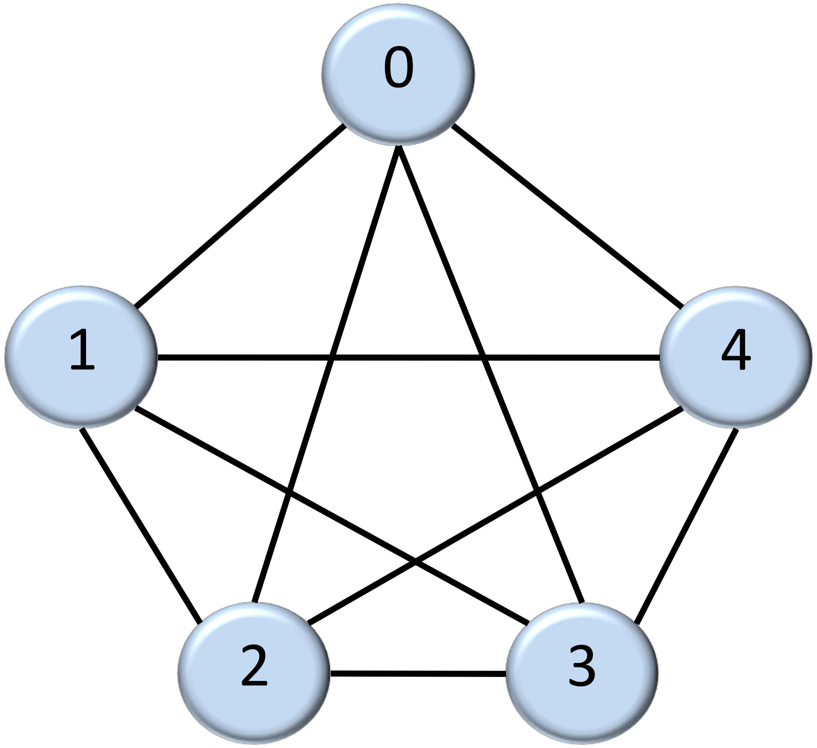
Round 1:
Broadcast own value

Round 2 to round $f+1$:
Broadcast the minimum of the received values
(unless it has been sent before)

End of round $f+1$:
Decide on the minimum value received

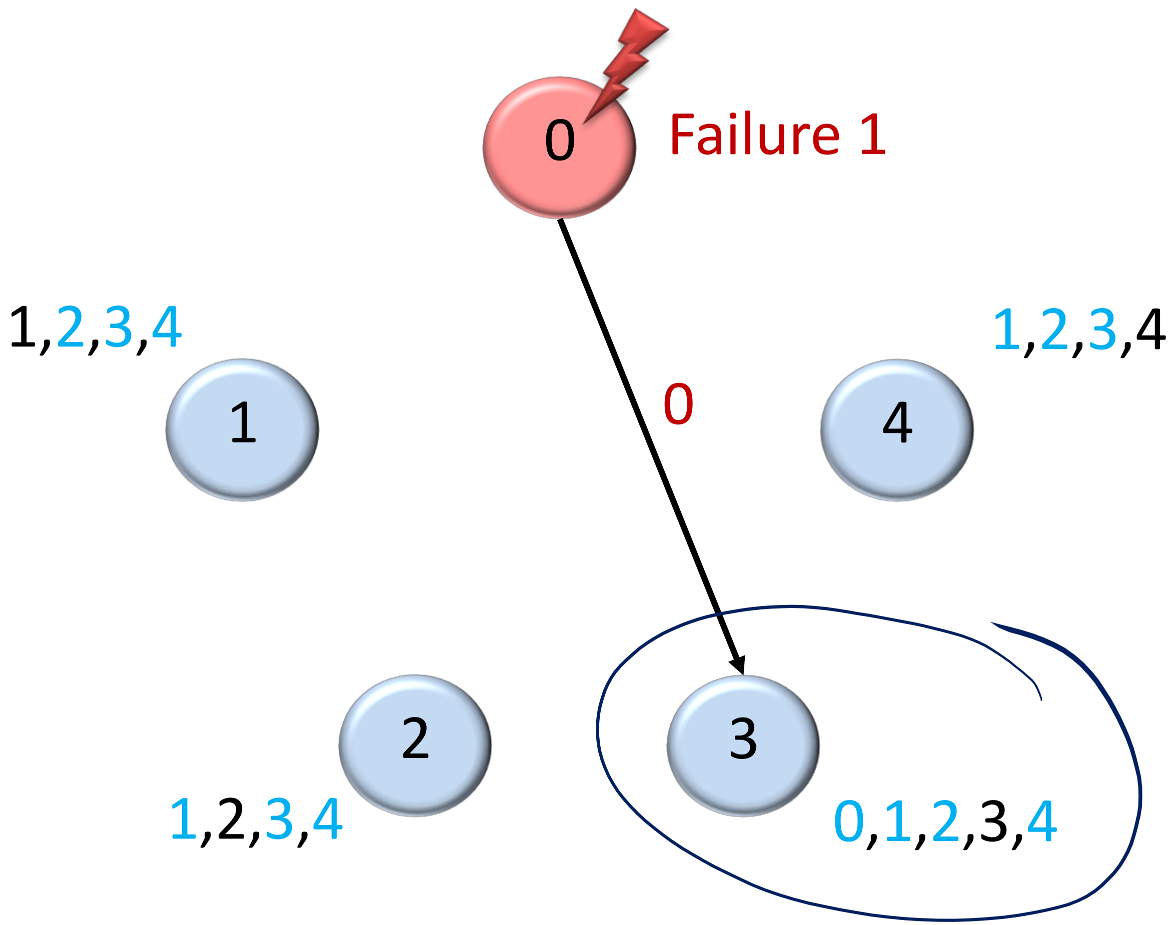
An f-Resilient Consensus Algorithm

- Example: $f = 2$ failures, $f + 1 = 3$ rounds needed



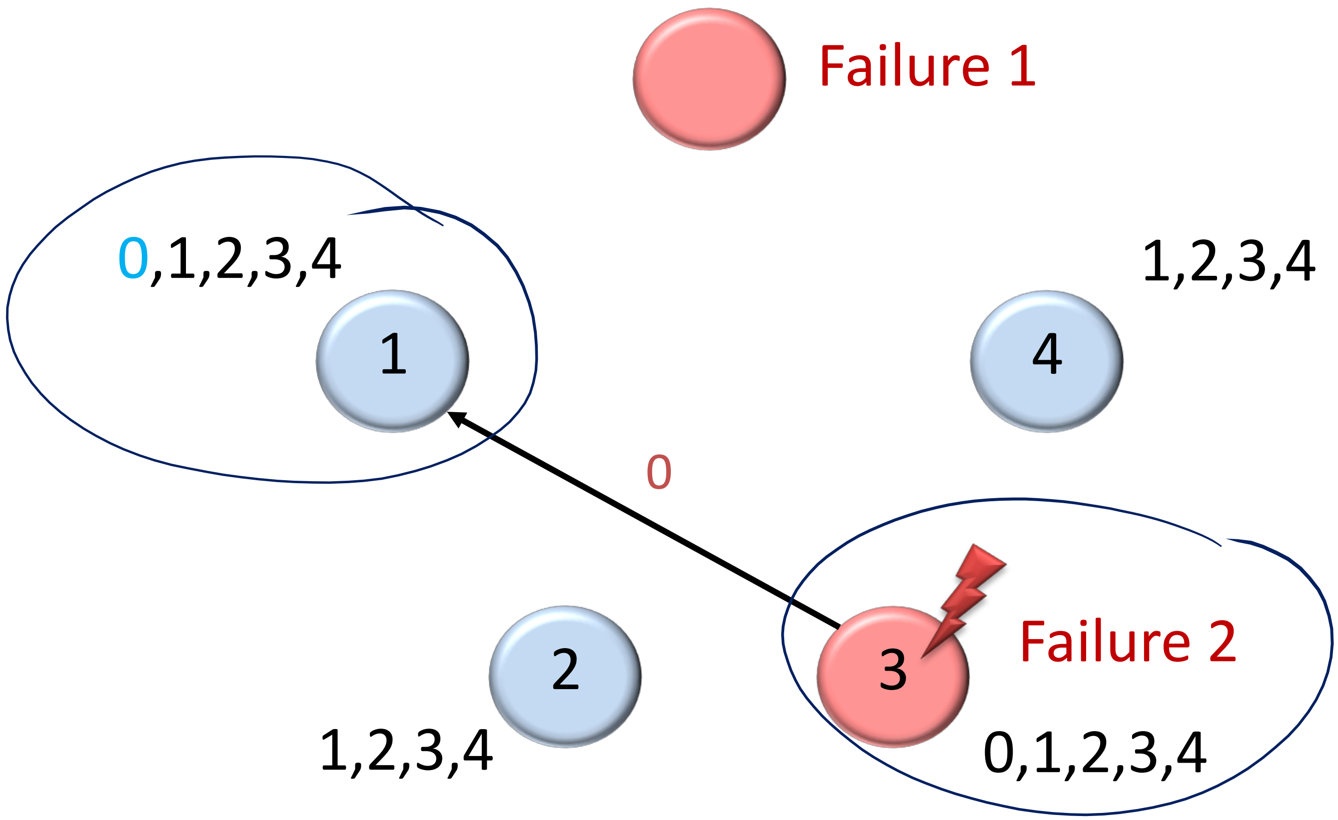
An f-Resilient Consensus Algorithm

- Round 1: Broadcast all values to everybody



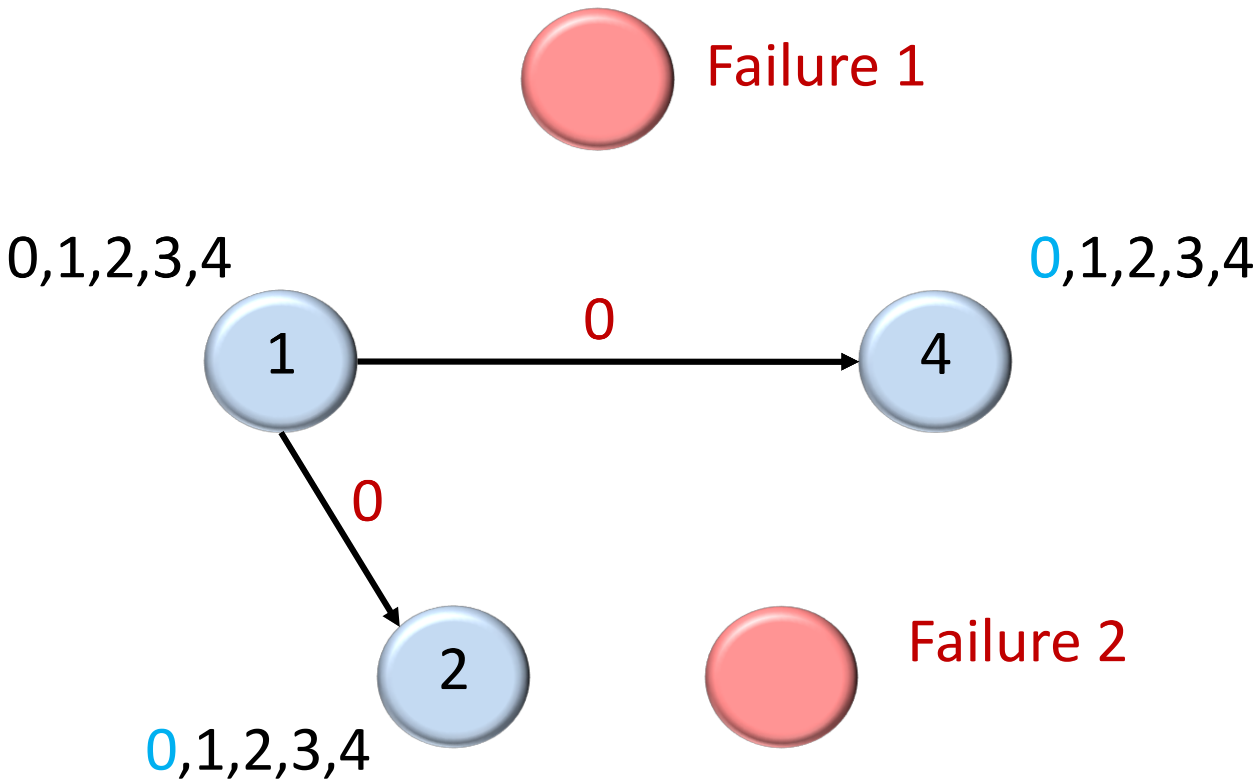
An f-Resilient Consensus Algorithm

- Round 2: Broadcast all new values to everybody



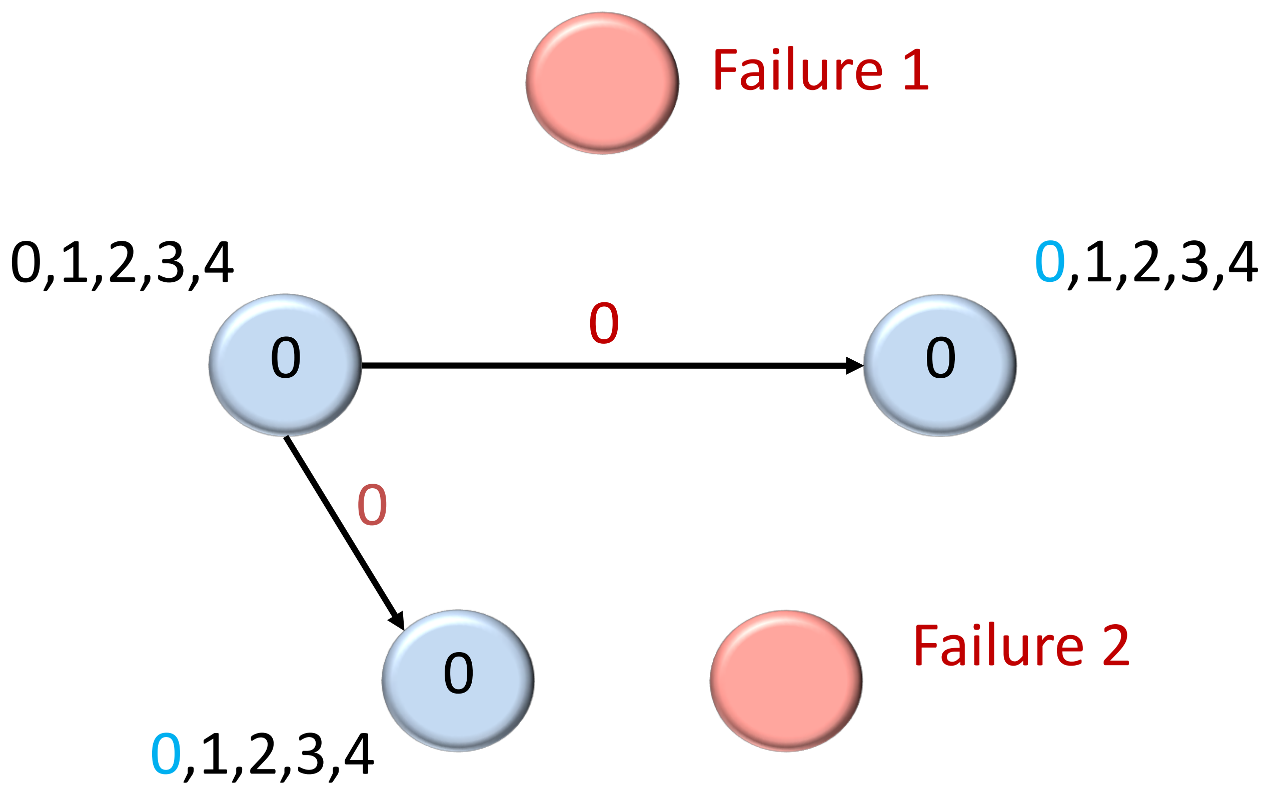
An f-Resilient Consensus Algorithm

- Round 3: Broadcast all new values to everybody



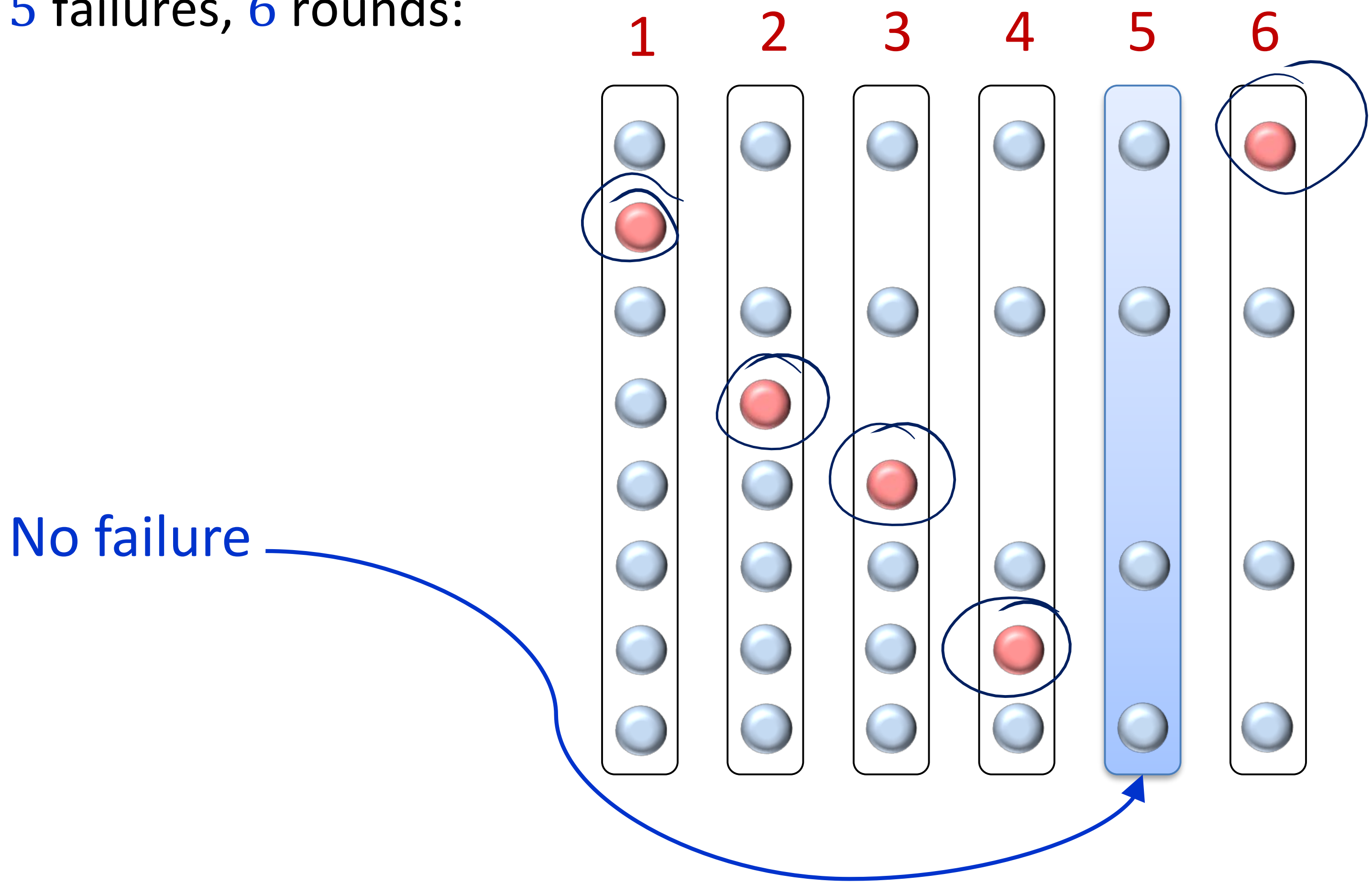
An f-Resilient Consensus Algorithm

- Decide on minimum → Consensus!



Analysis

- If there are f failures and $f+1$ rounds, then there is a round with no failed process
- Example: 5 failures, 6 rounds:



Analysis

- At the end of the round with no failure
  - Every (non faulty) process knows about all the values of all the other participating processes
  - This knowledge doesn't change until the end of the algorithm
- Therefore, everybody will decide on the same value
- However, as we don't know the exact position of this round, we have to let the algorithm execute for $f + 1$ rounds

- **Validity:** When all processes start with the same input value, then consensus is that value

Theorem

Theorem

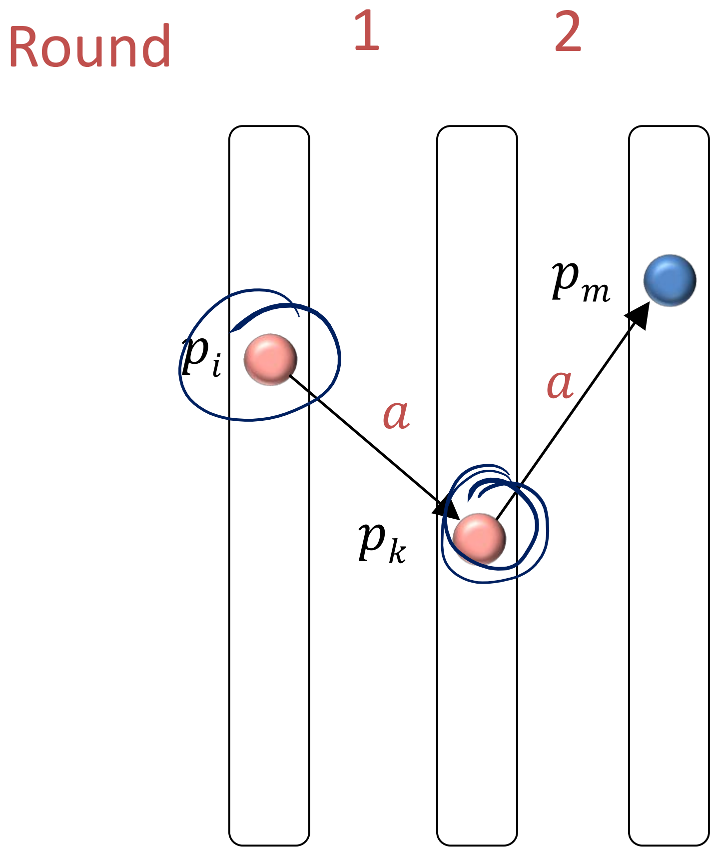
If at most $f \leq n-2$ of n nodes of a synchronous message passing system can crash, at least $f+1$ rounds are needed to solve consensus.

Proof idea:

- Show that f rounds are not enough if $n \geq f+2$
- Before proving the theorem, we consider a

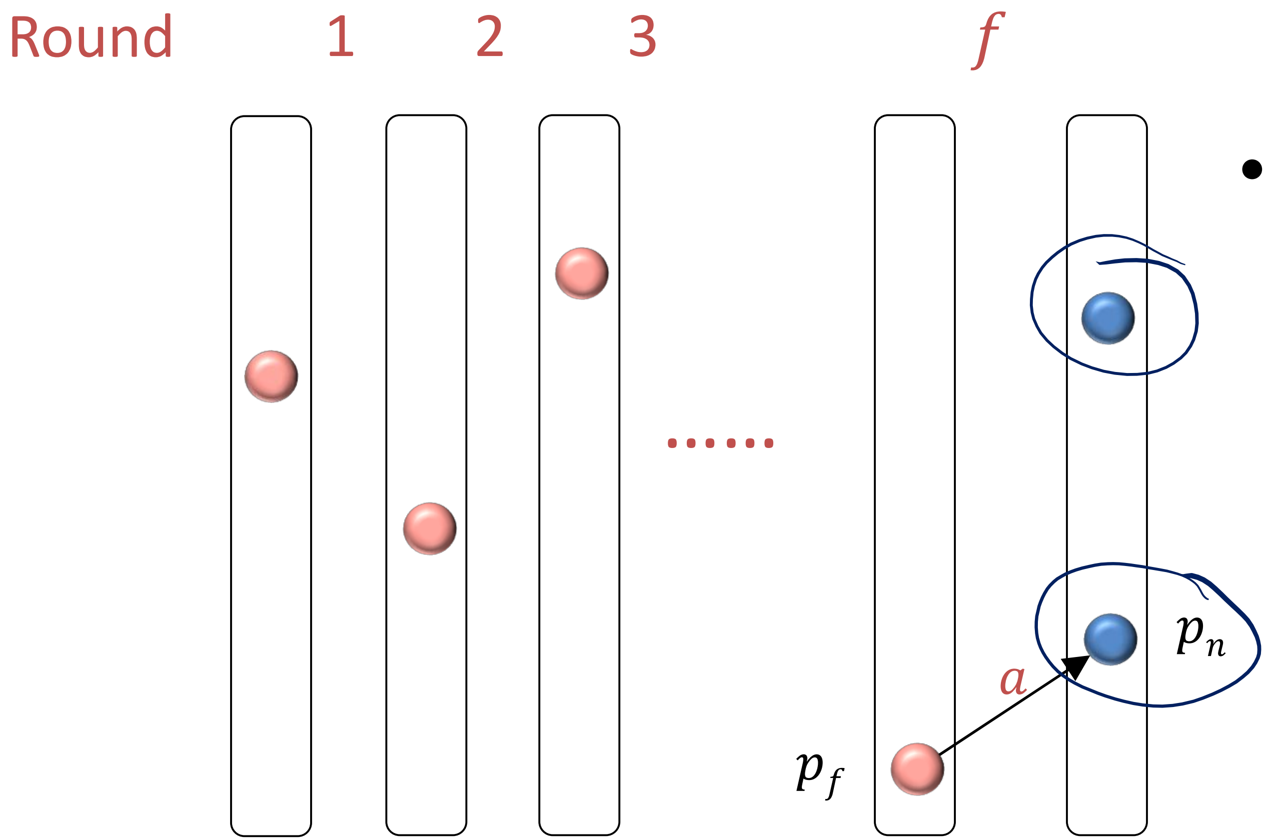
  "worst-case scenario": In each round one of the processes fails

Lower Bound on Rounds: Intuition



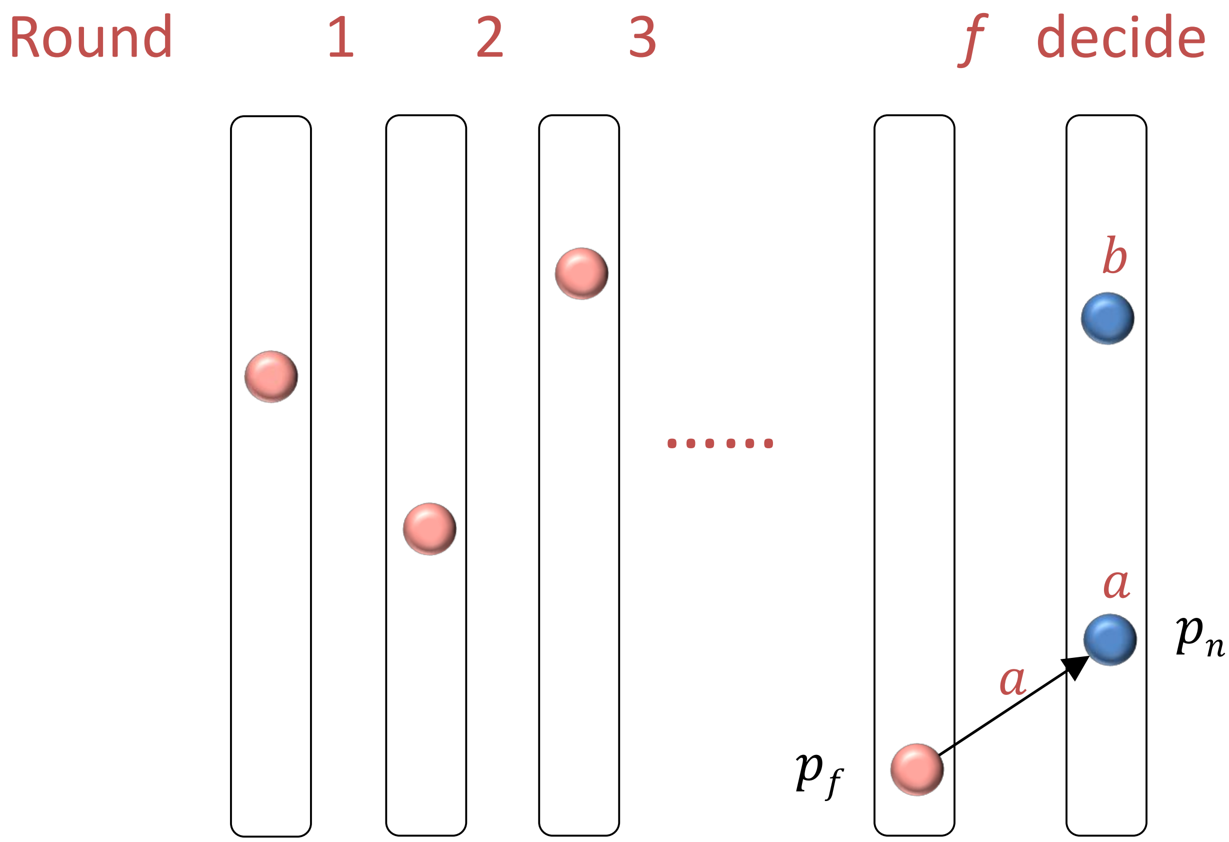
- Before process p_i fails, it sends its value a only to one process p_k
- Before process p_k fails, it sends its value a to only one process p_m

Lower Bound on Rounds: Intuition



- At the end of round f only one process p_n knows about value a

Lower Bound on Rounds: Intuition



- Process p_n may decide on a and all other processes may decide on another value b
- f rounds are not enough
  $\Longrightarrow$ at least $f+1$ rounds are needed

Lower Bound on Rounds: Proof

Recall from earlier in the course:

- For the impossibility proof of the two generals problem, we used an indistinguishability proof
- Execution E is indistinguishable from execution E' for some node v if v sees the same things in both executions.
  - same inputs and messages (schedule)
- If E is indistinguishable from E' for v, then v does the same thing in both executions.
  - We denoted this by $E|v = E'|v$

Similarity:

- Call E_i and E_j **similar** if $E_i|v = E_j|v$ for some node v

$$E_i \sim_v E_j \;\Leftrightarrow\; E_i|v = E_j|v$$

$$E_i \sim E_j \Leftrightarrow \exists v : E_i \sim_v E_j$$

Lower Bound on Rounds: Proof

Similarity Chain:

- Consider a sequence of executions $E_1, E_2, E_3, \dots, E_T$ such that

$$\forall i \geq 1 \;:\quad E_i \sim_{v_i} E_{i+1}$$

  - any two consecutive executions E_i and E_{i+1} are indistinguishable for some node v_i (we assume that v_i does not crash in E_i and E_{i+1})

- **Indistinguishability:**
  $\forall i \geq 1 \;:\;$ Node v_i decides on the same value in E_i and E_{i+1}

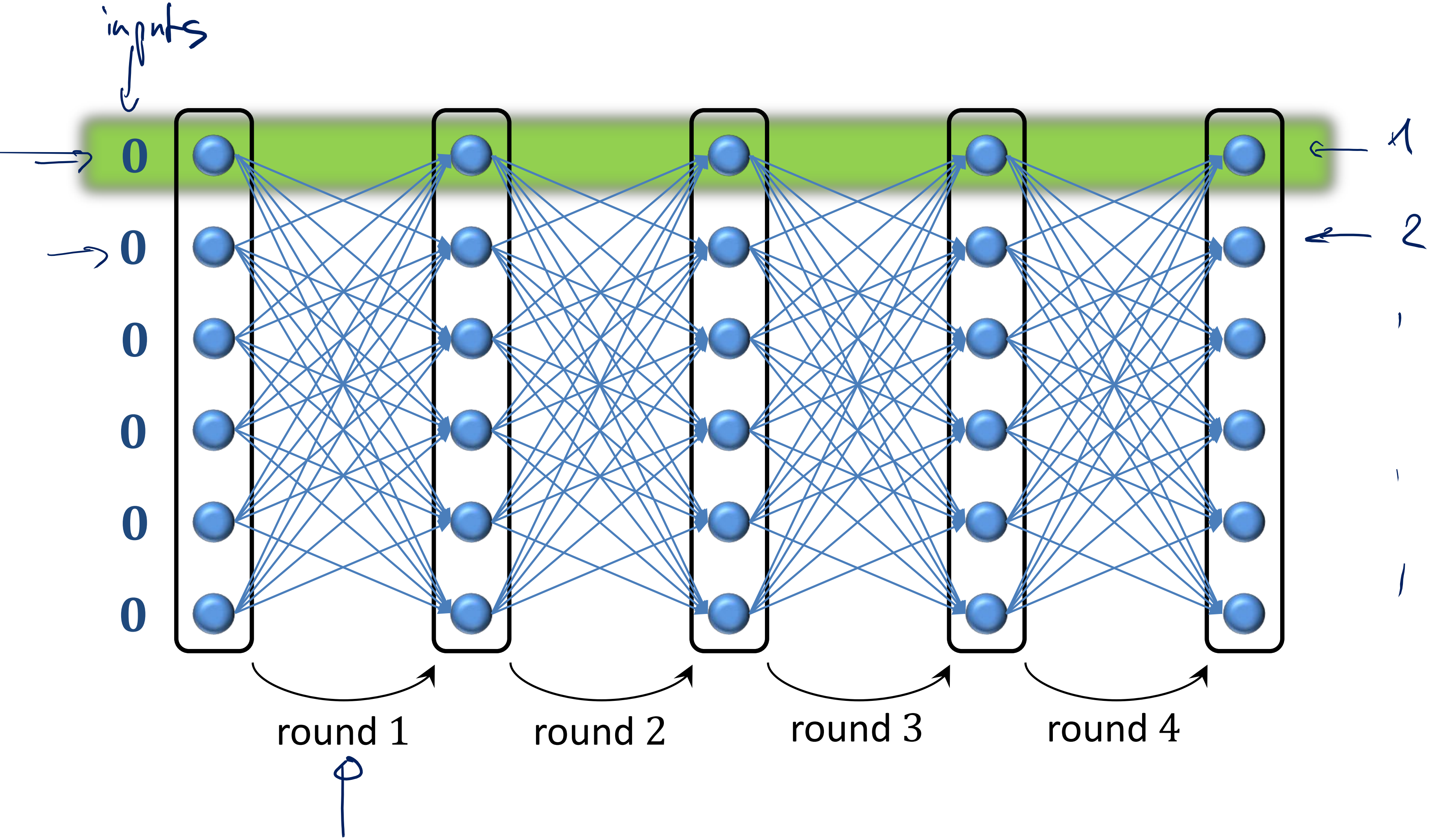
- **Agreement:**
  $\forall i \geq 1 \;:\;$ All nodes decide on the same value in E_i and E_{i+1}

- Hence, all executions $E_1, \dots, E_T$ have the same decision value!

- **Goal:**
  $\underbrace{E_1\text{: no crashes, all inputs are } 0}_{0}$; $\underbrace{E_T\text{: no crashes, all inputs are } 1}_{1}$

Lower Bound on Rounds: Proof

Example: $f = 4, n = 6$ **Need to show: 4 rounds are not enough**



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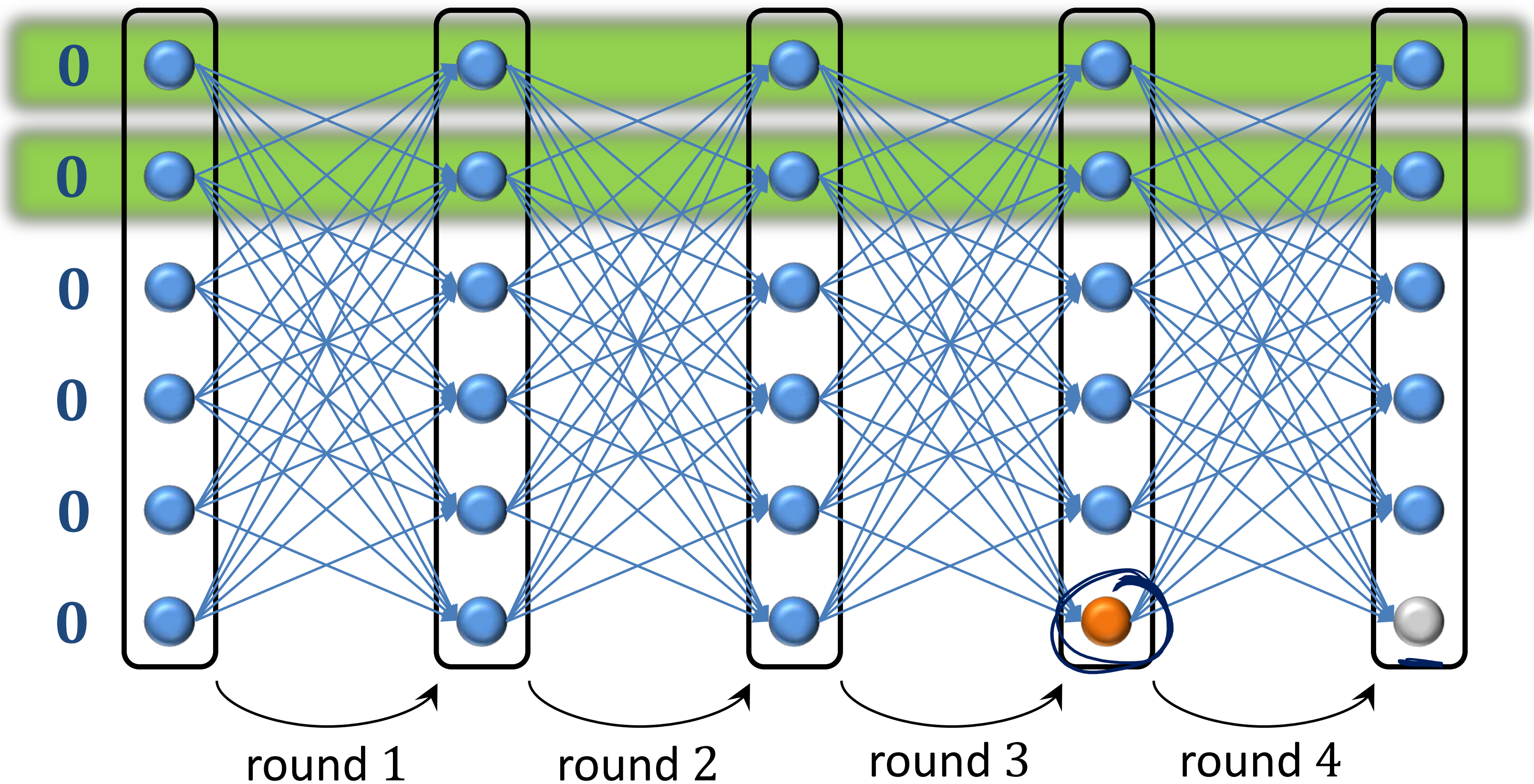

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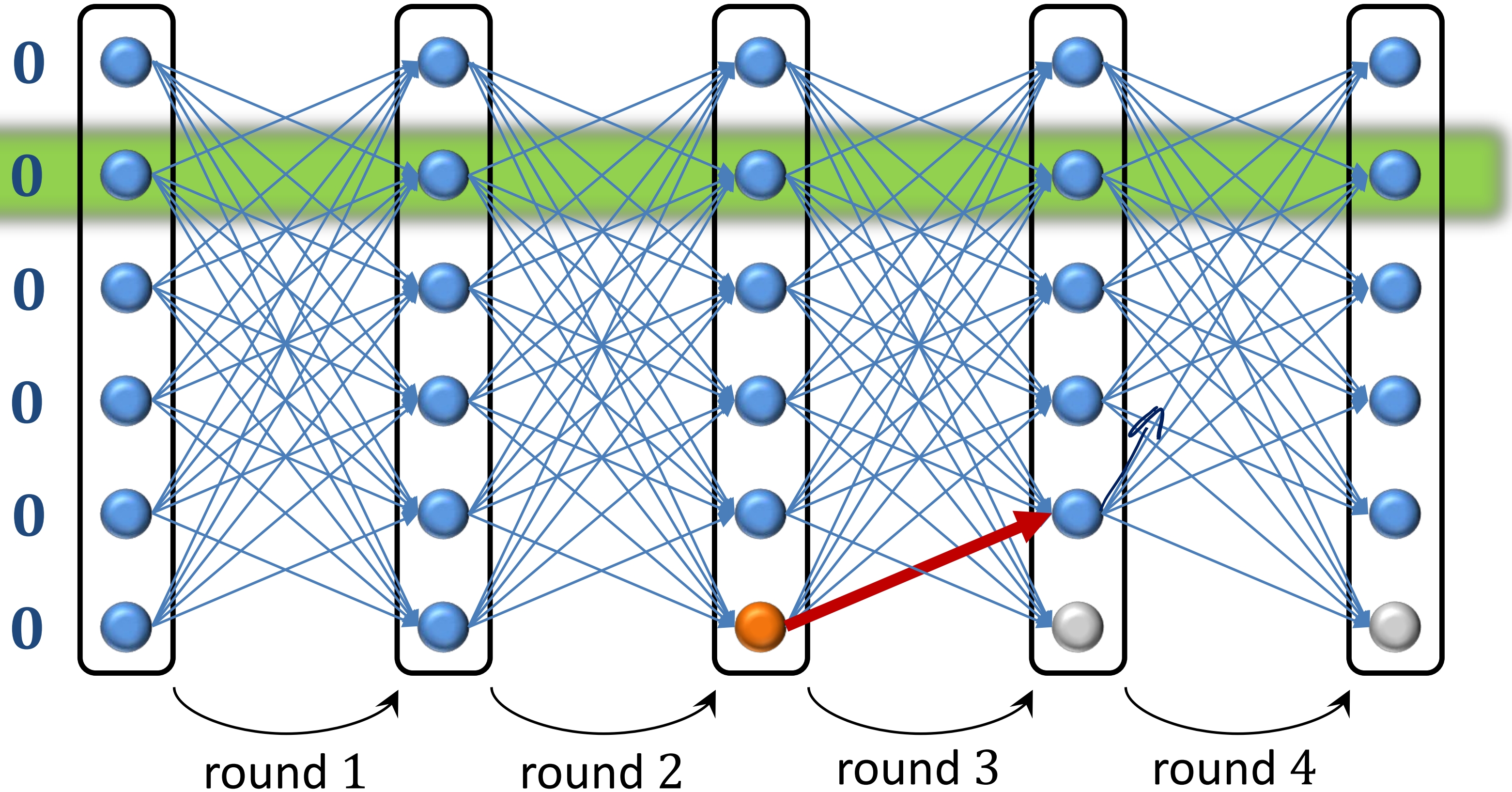

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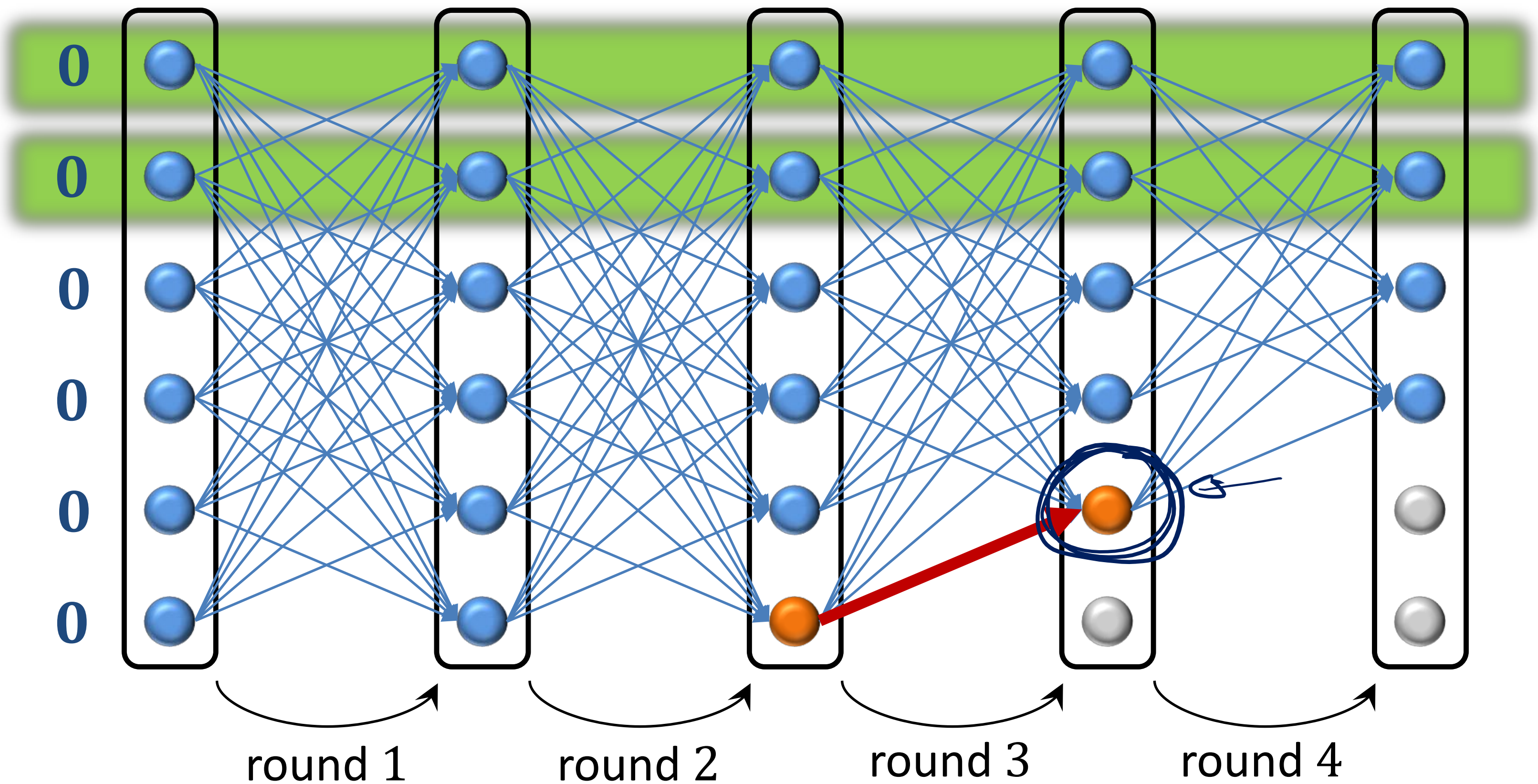

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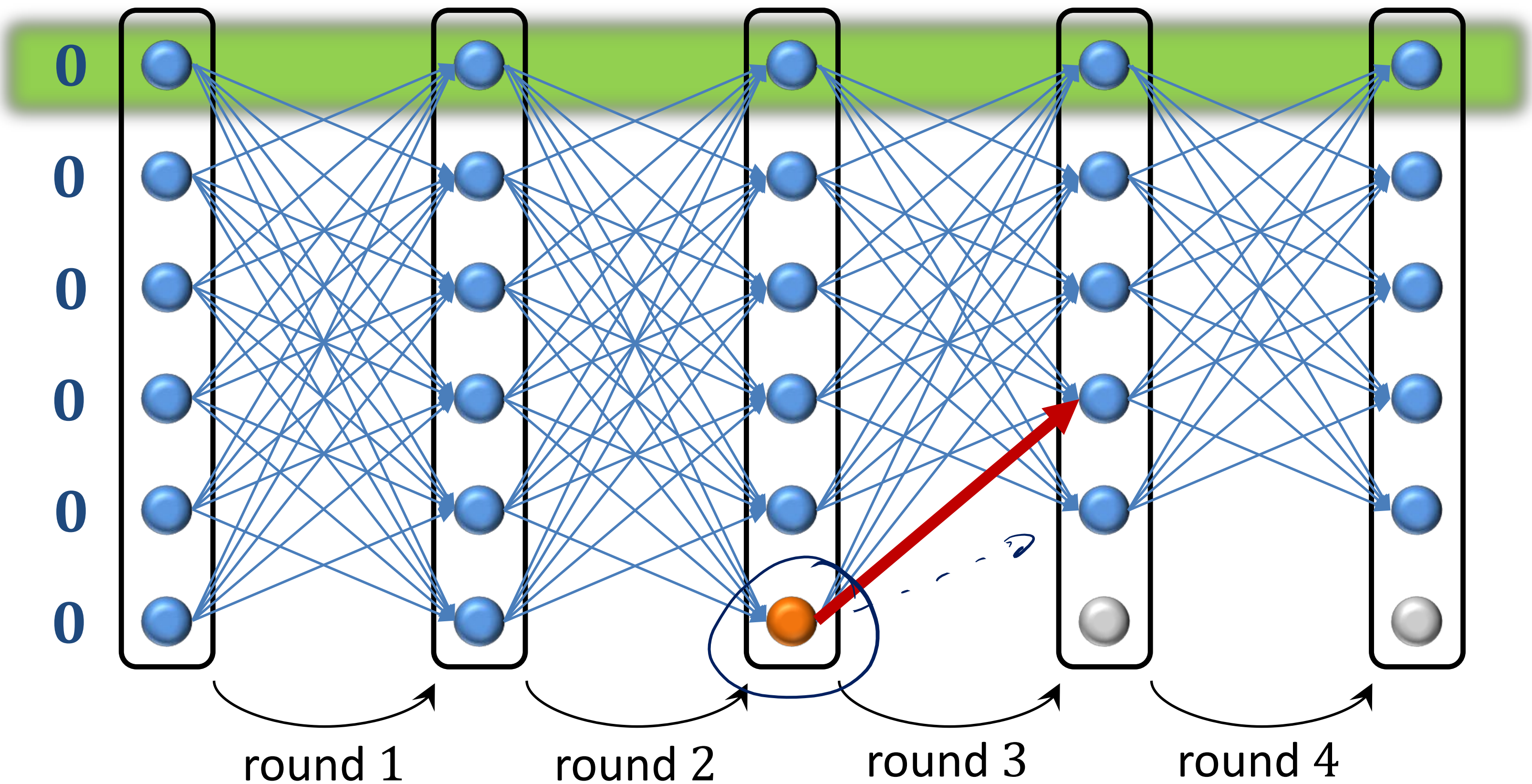

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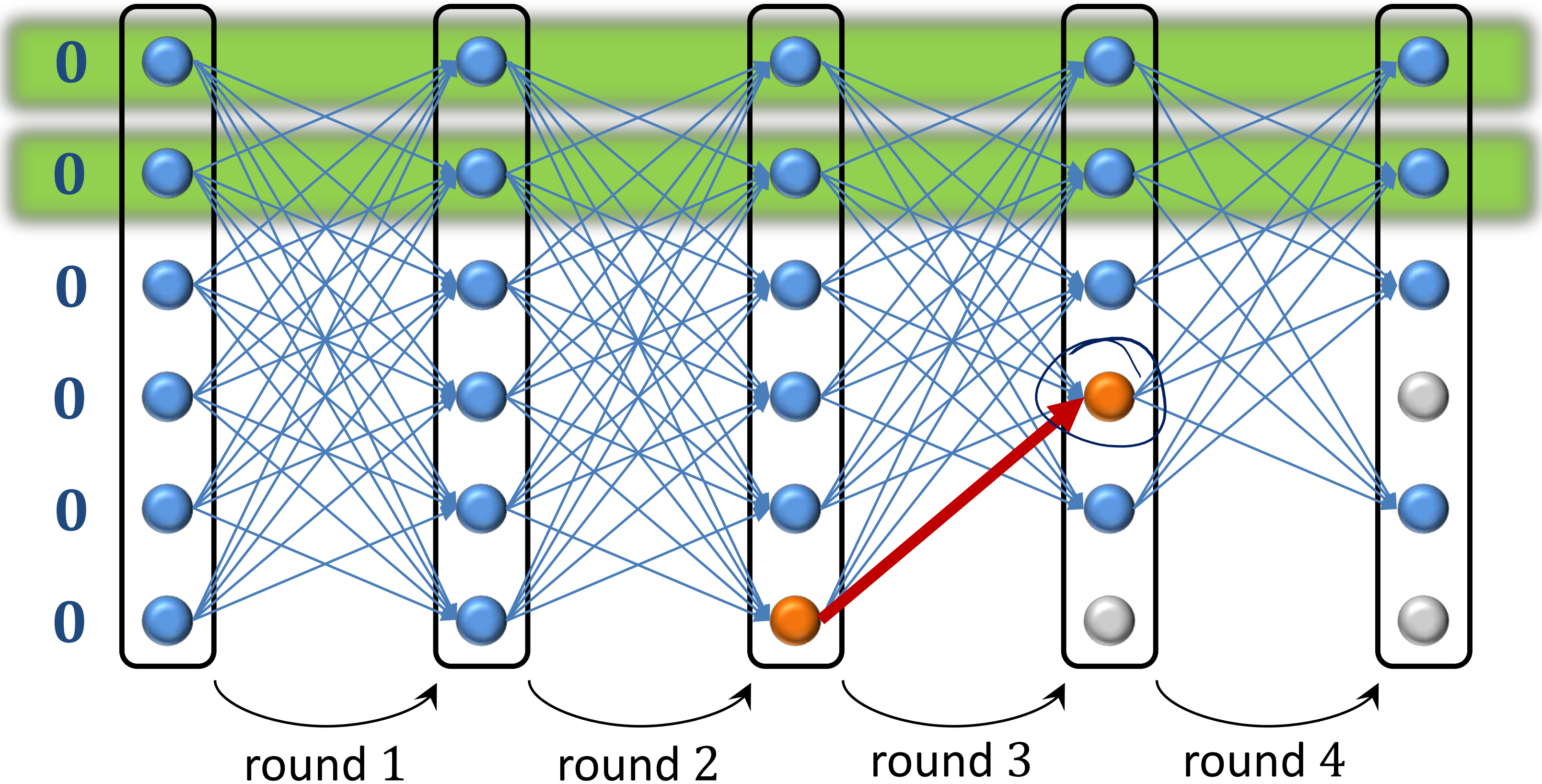

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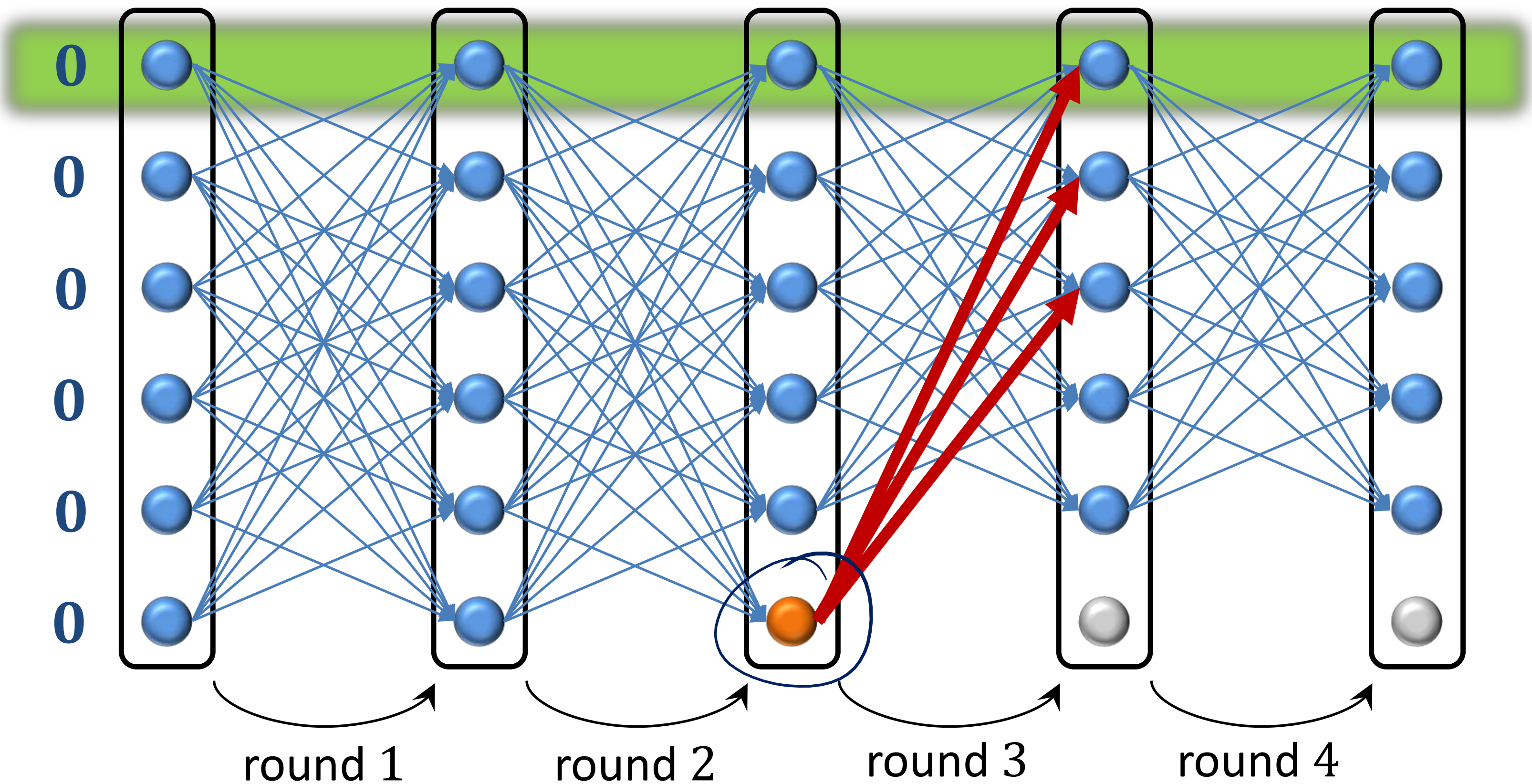

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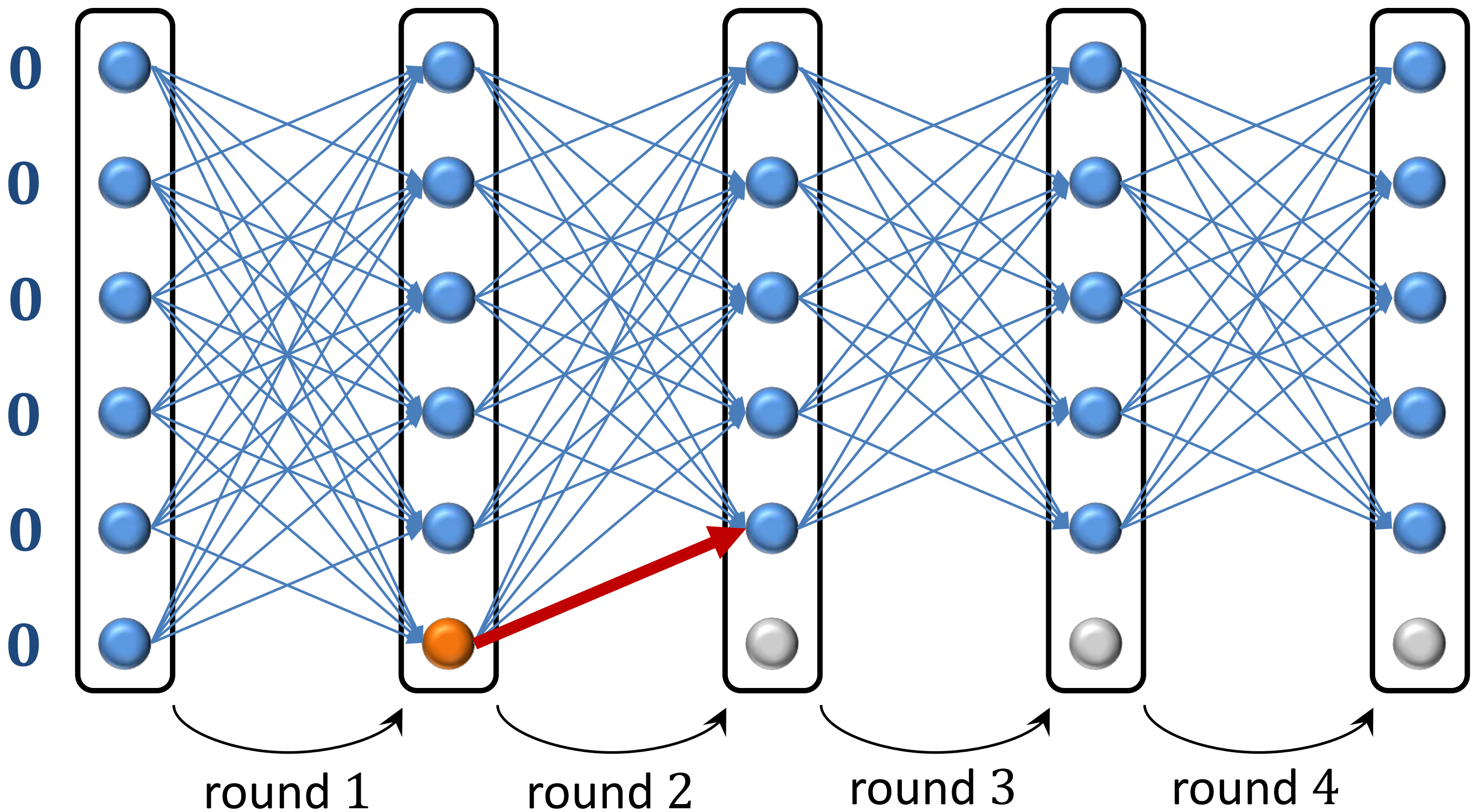

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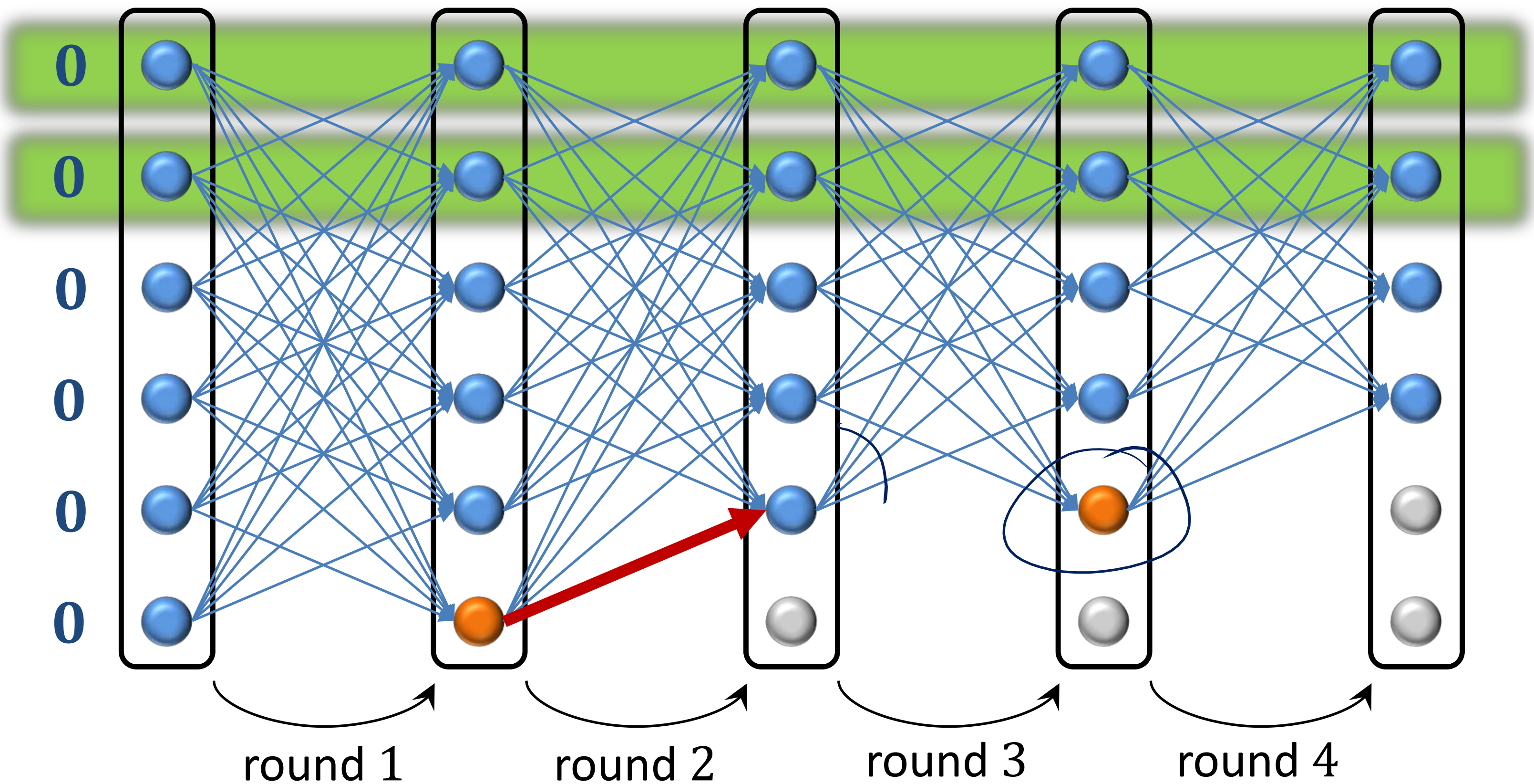

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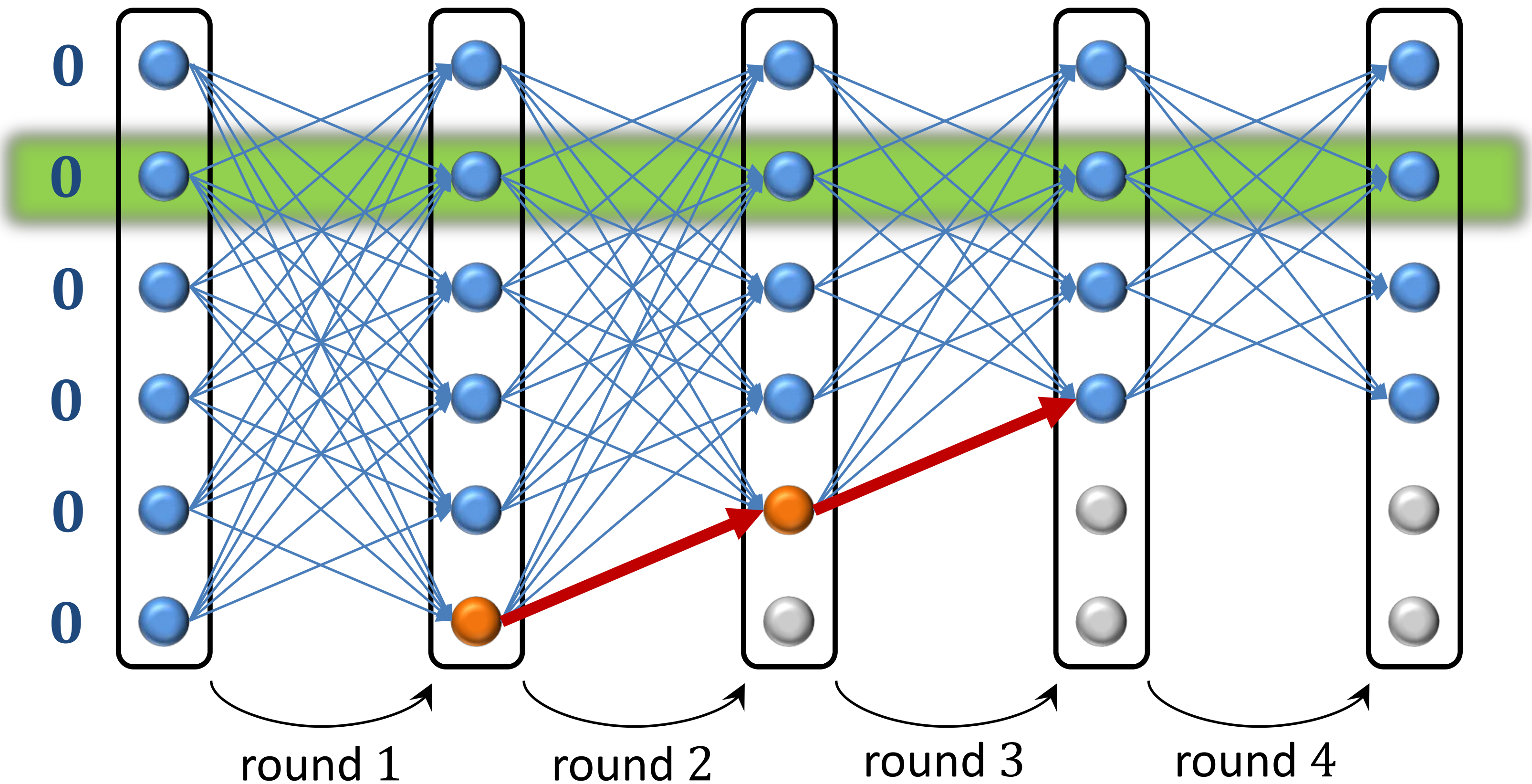

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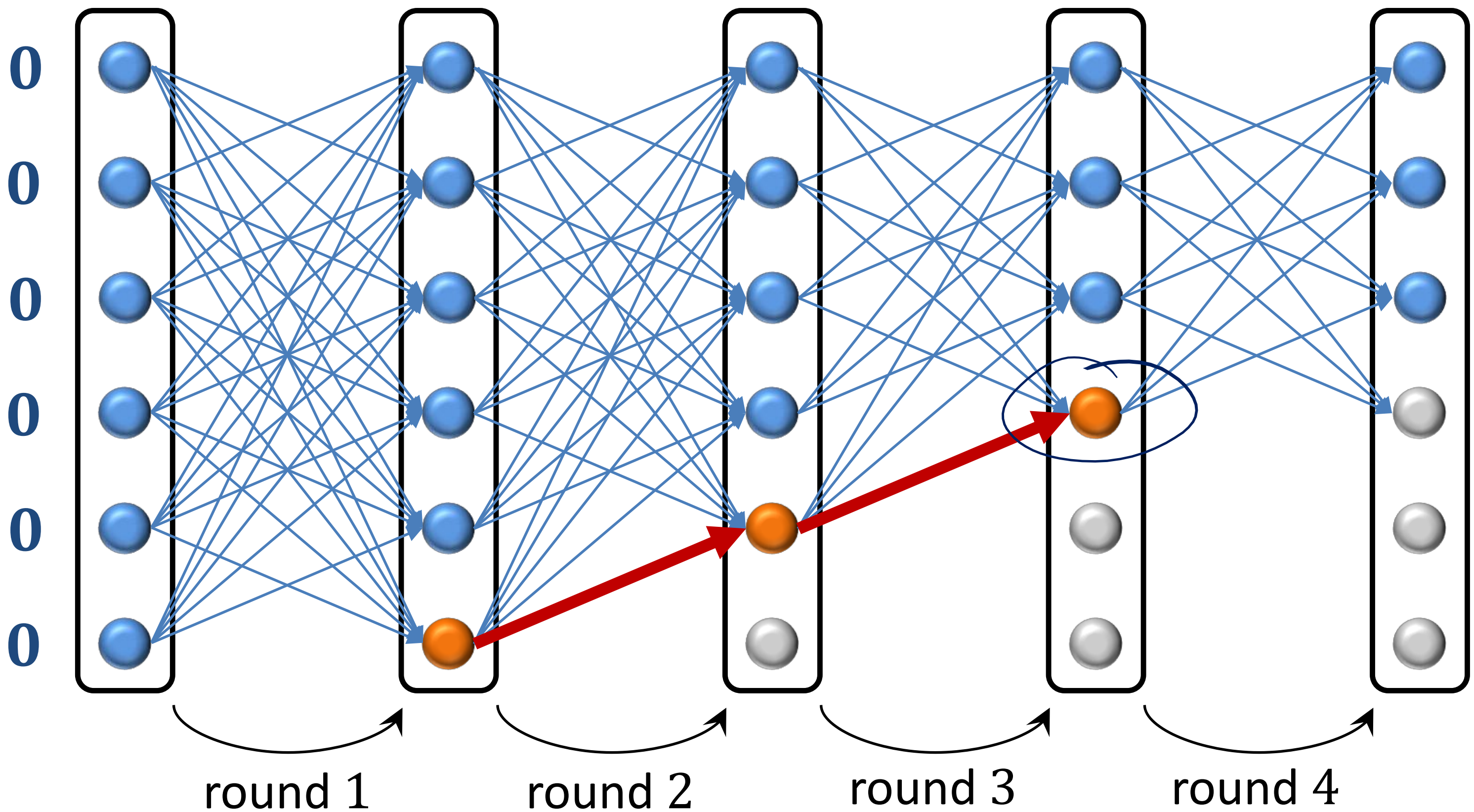

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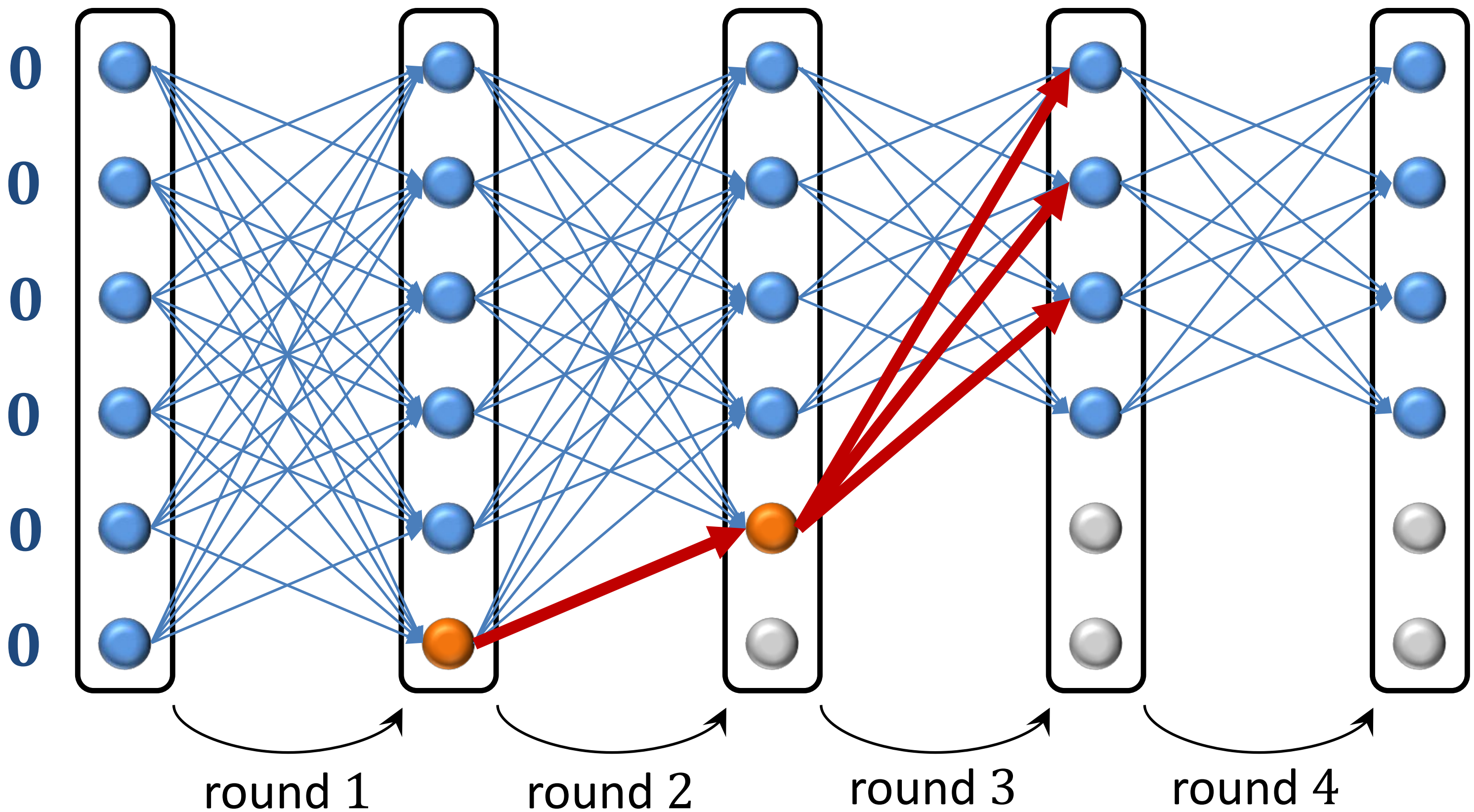

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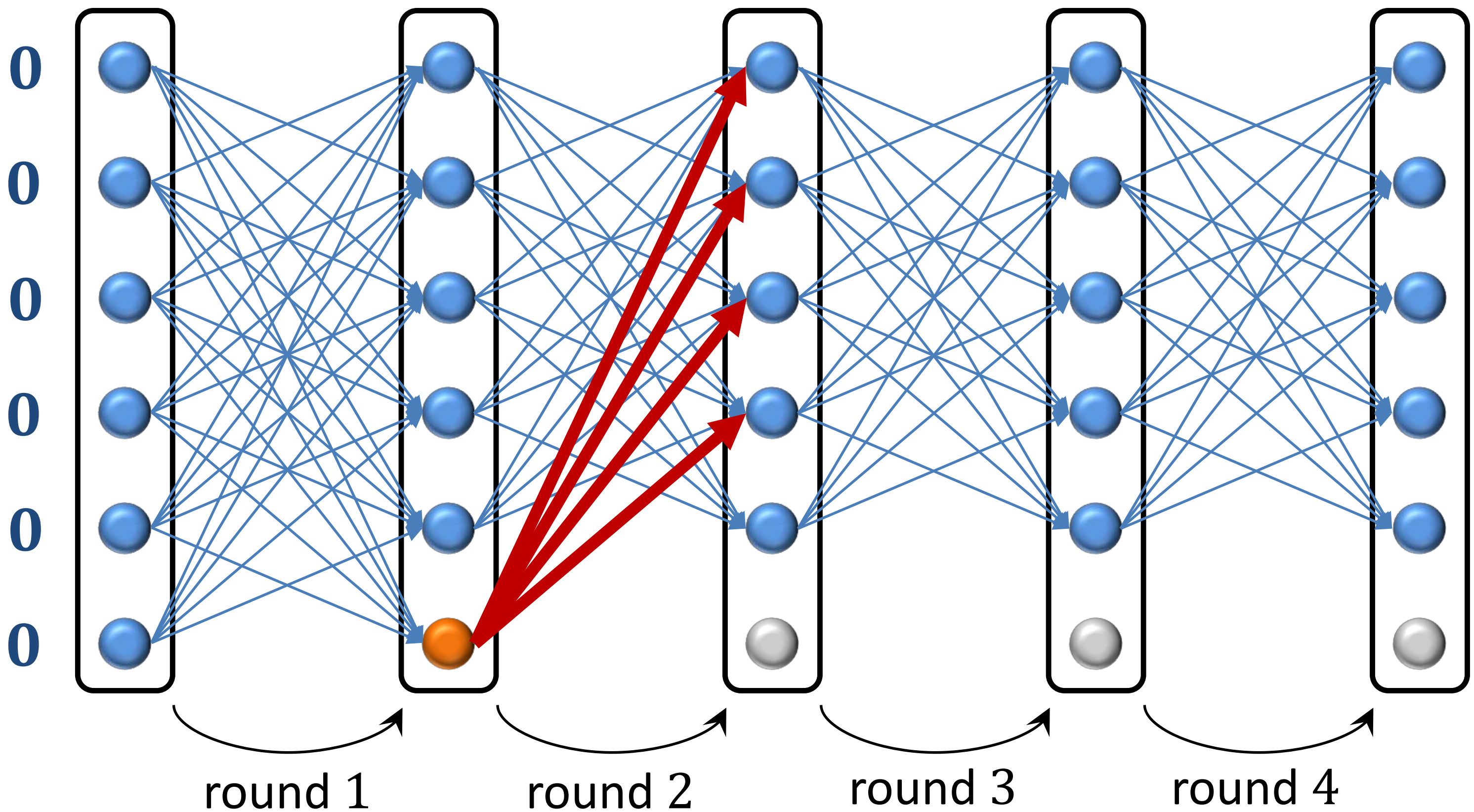

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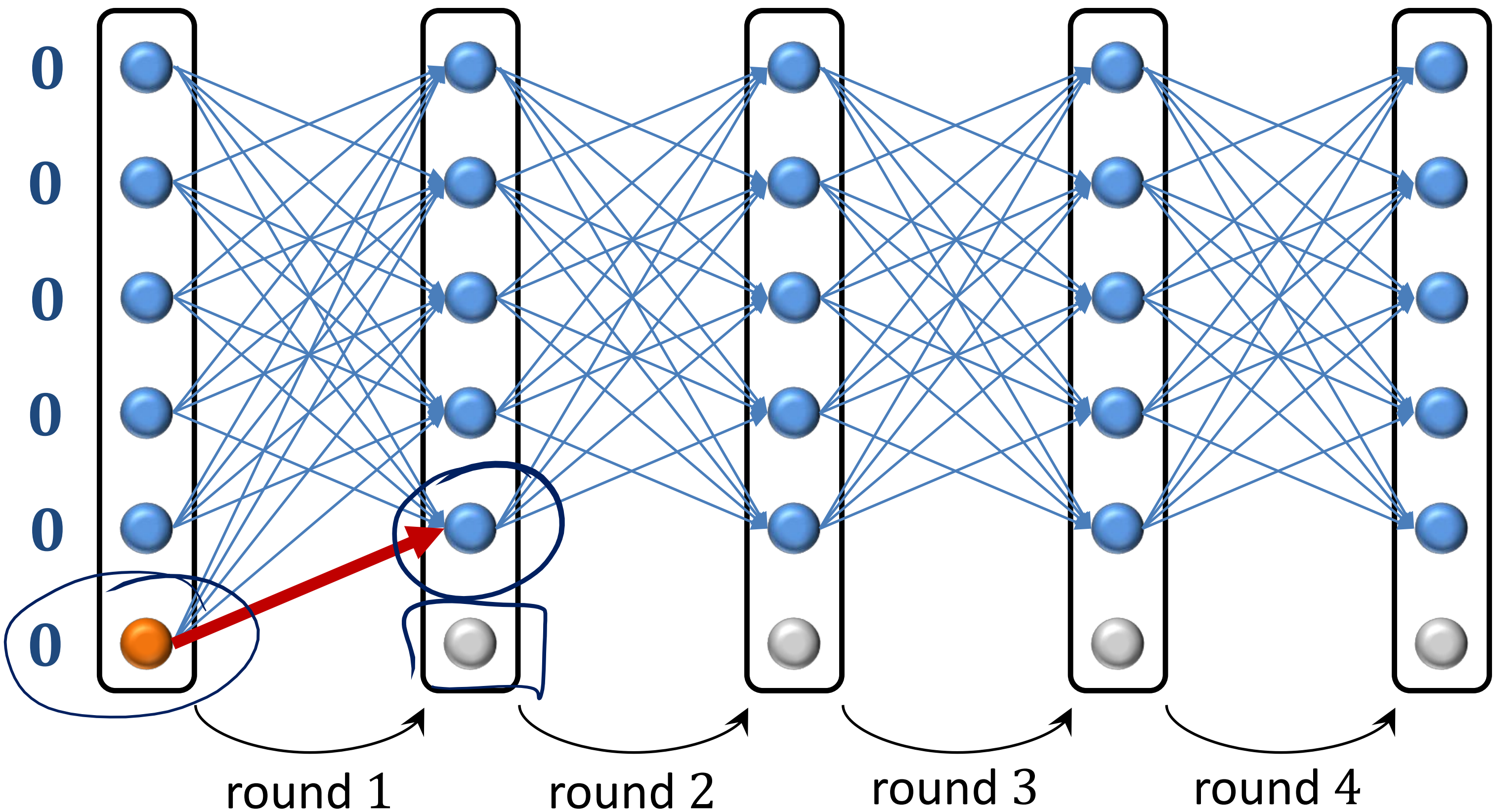

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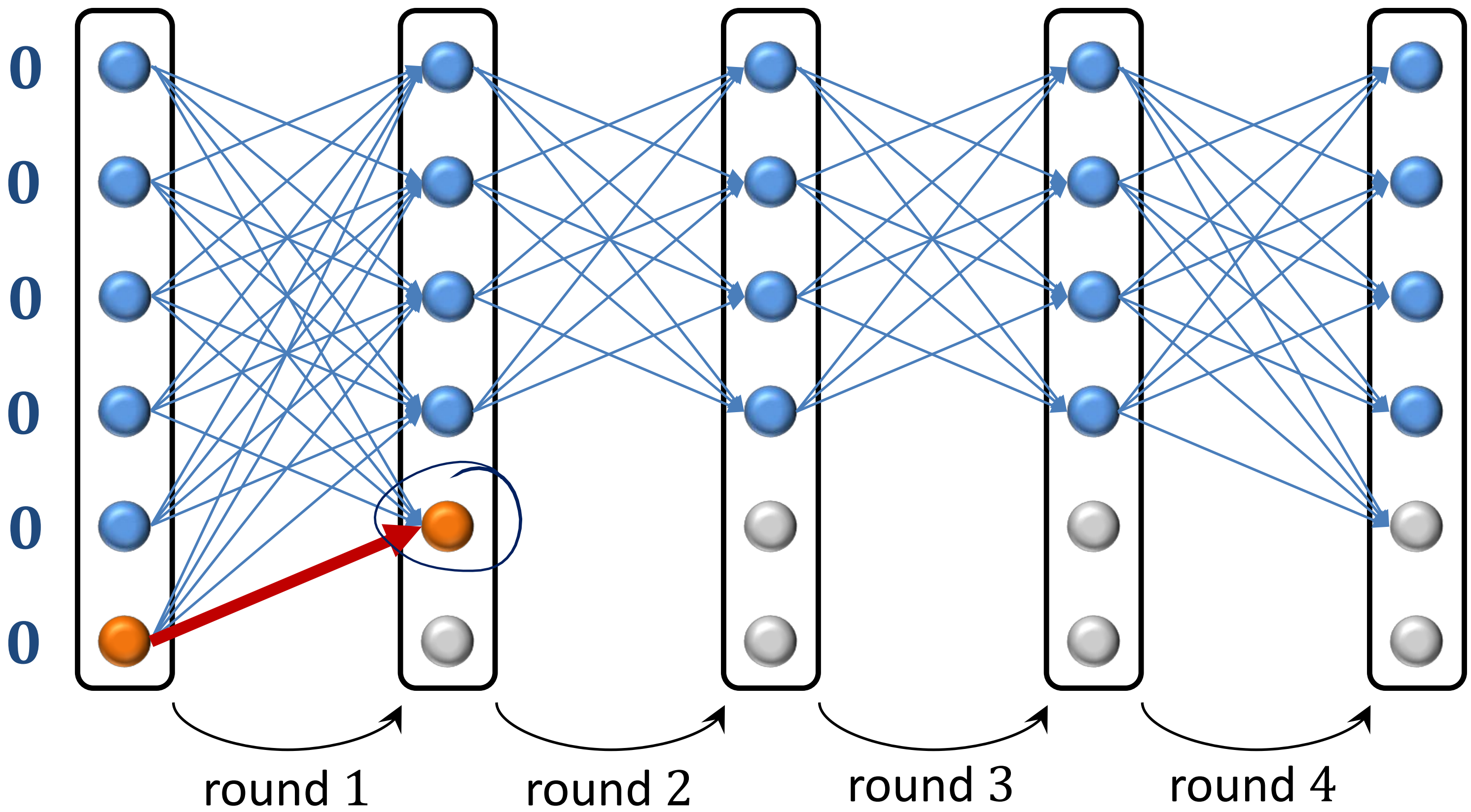

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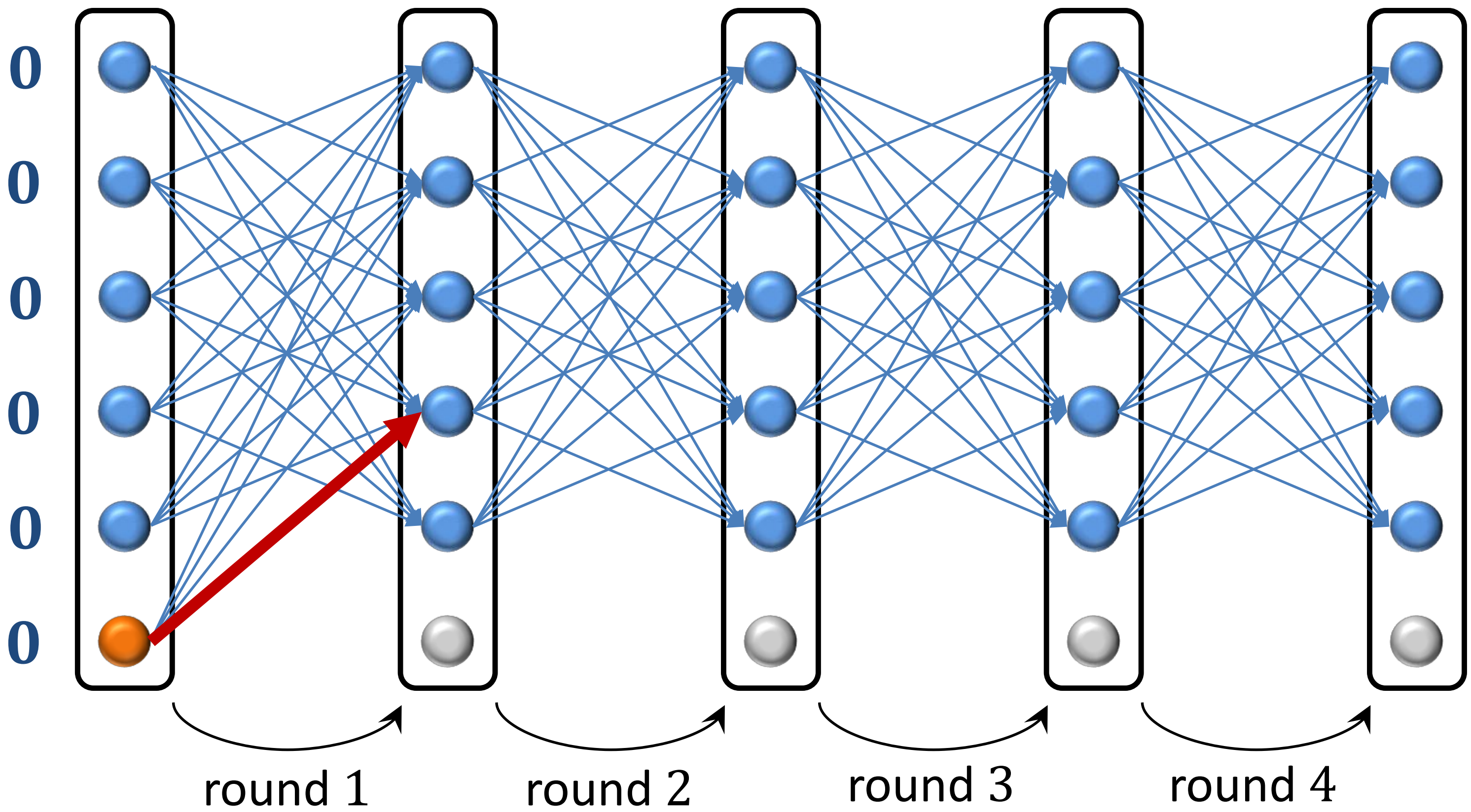

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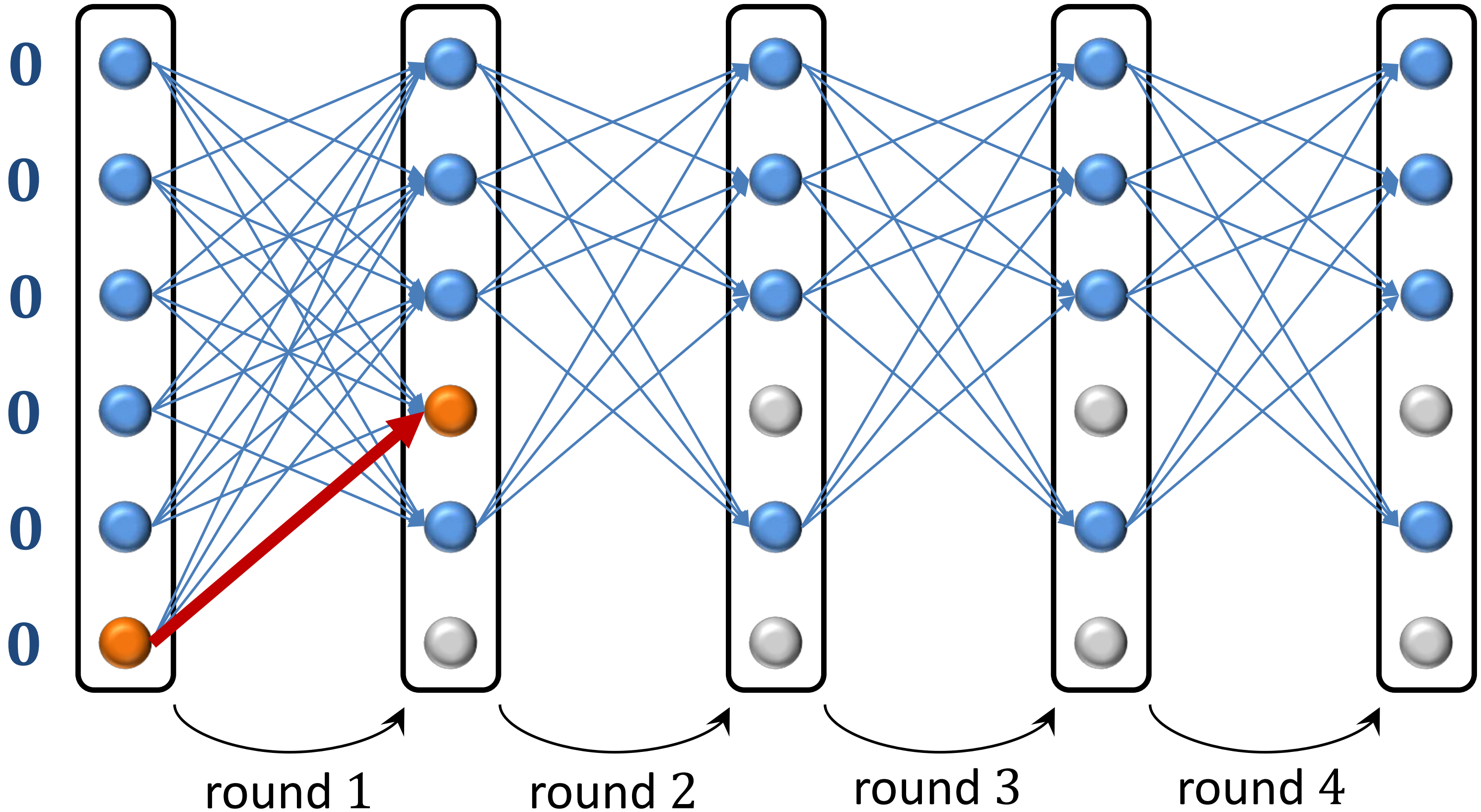

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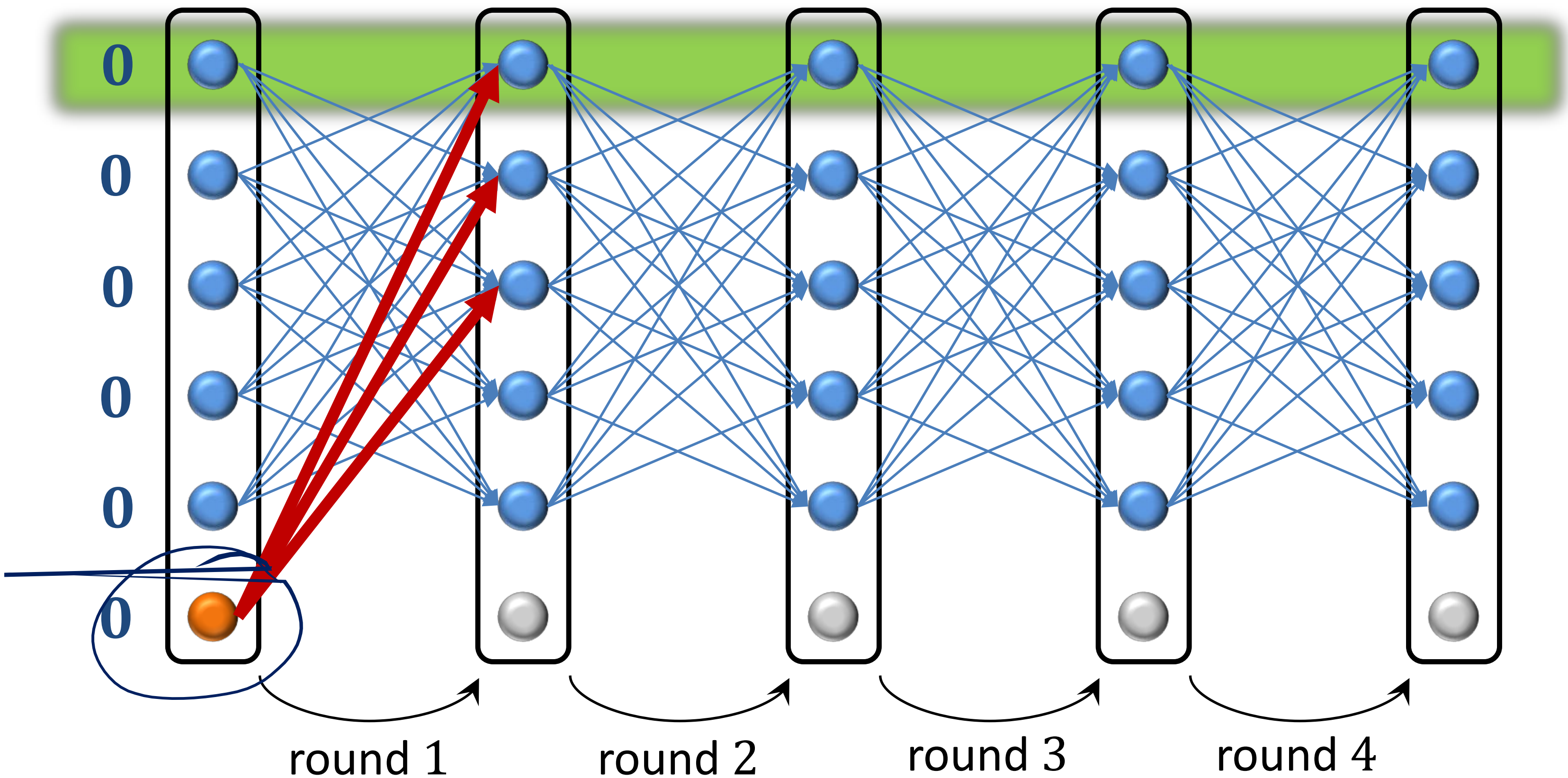

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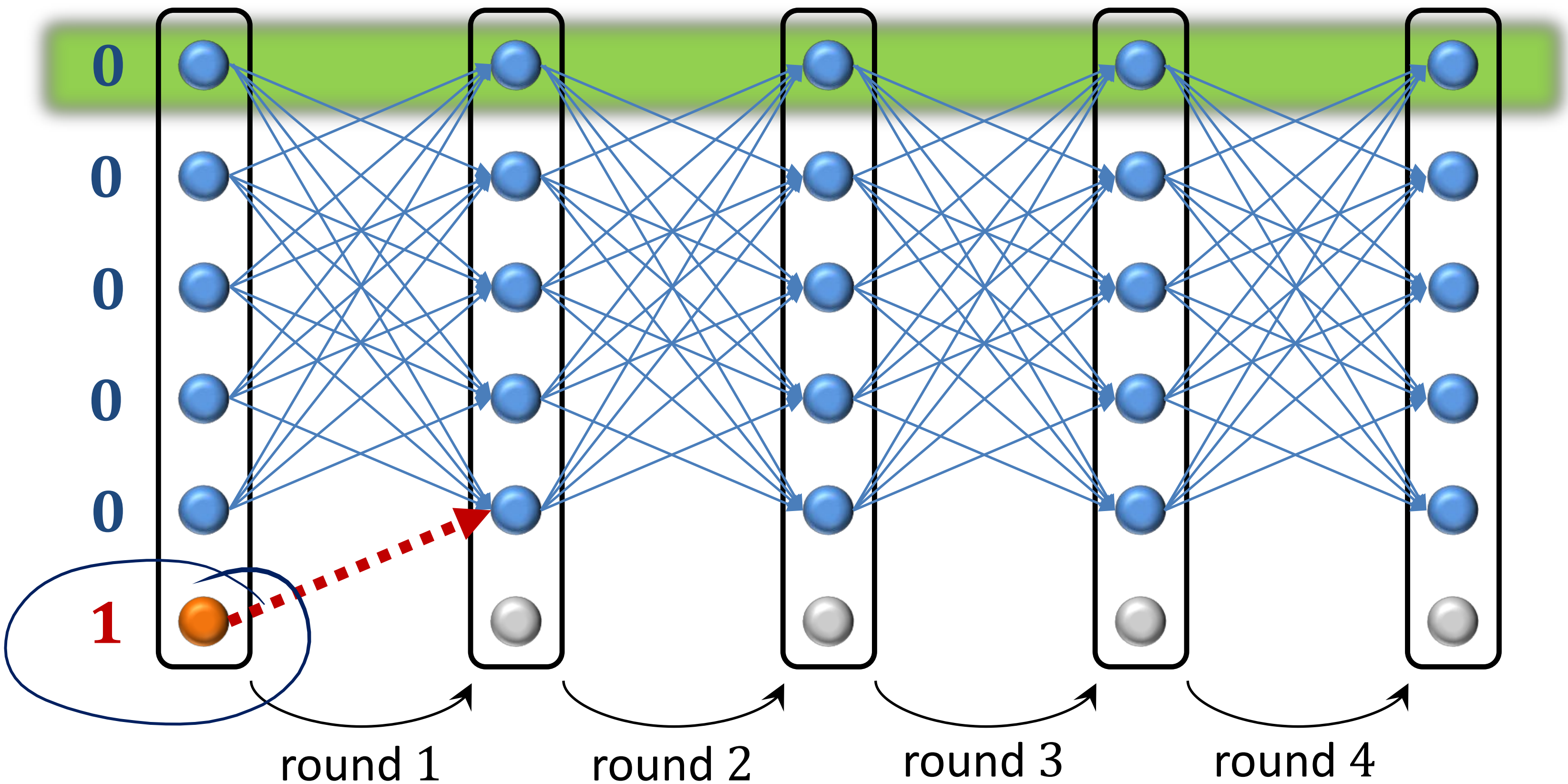

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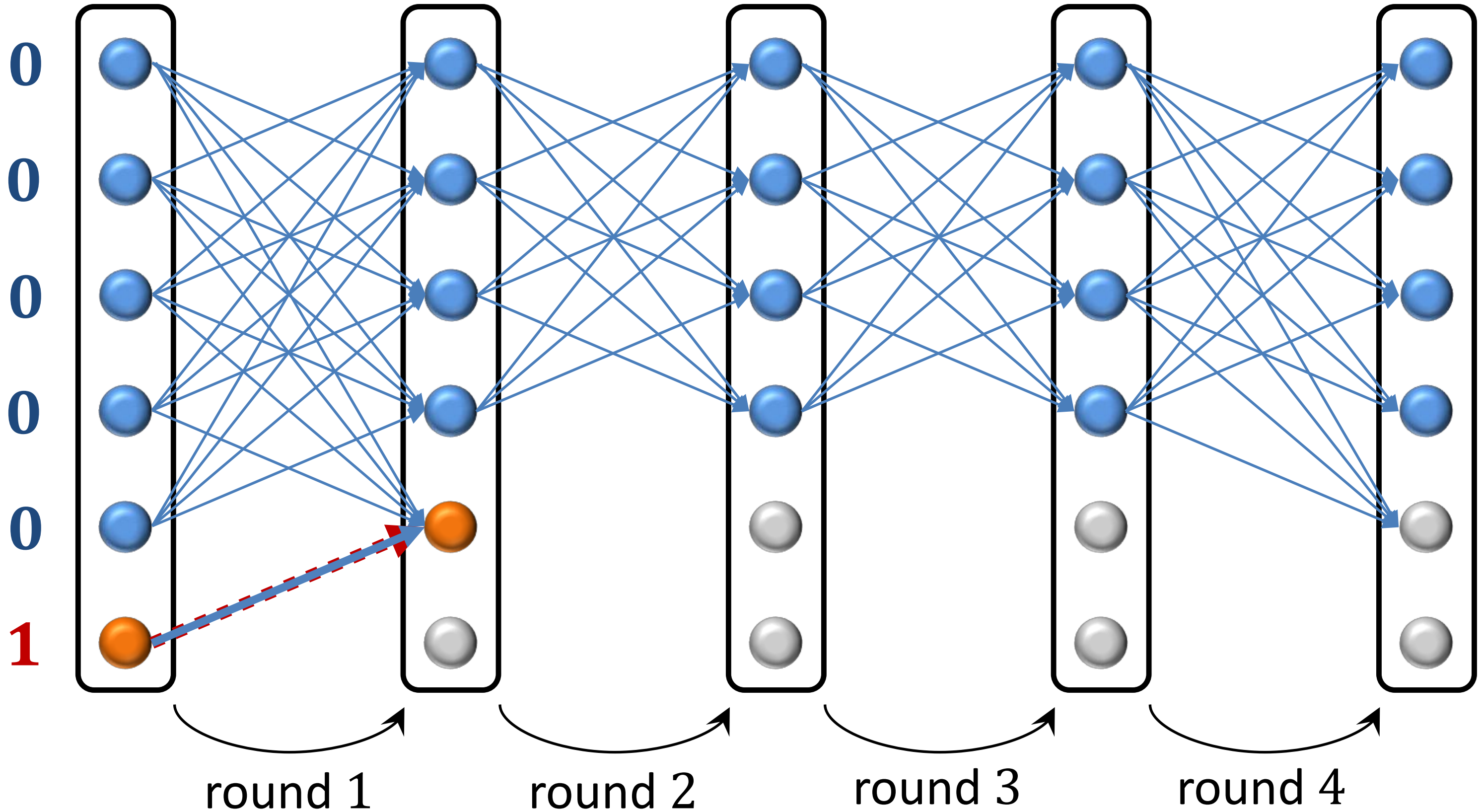

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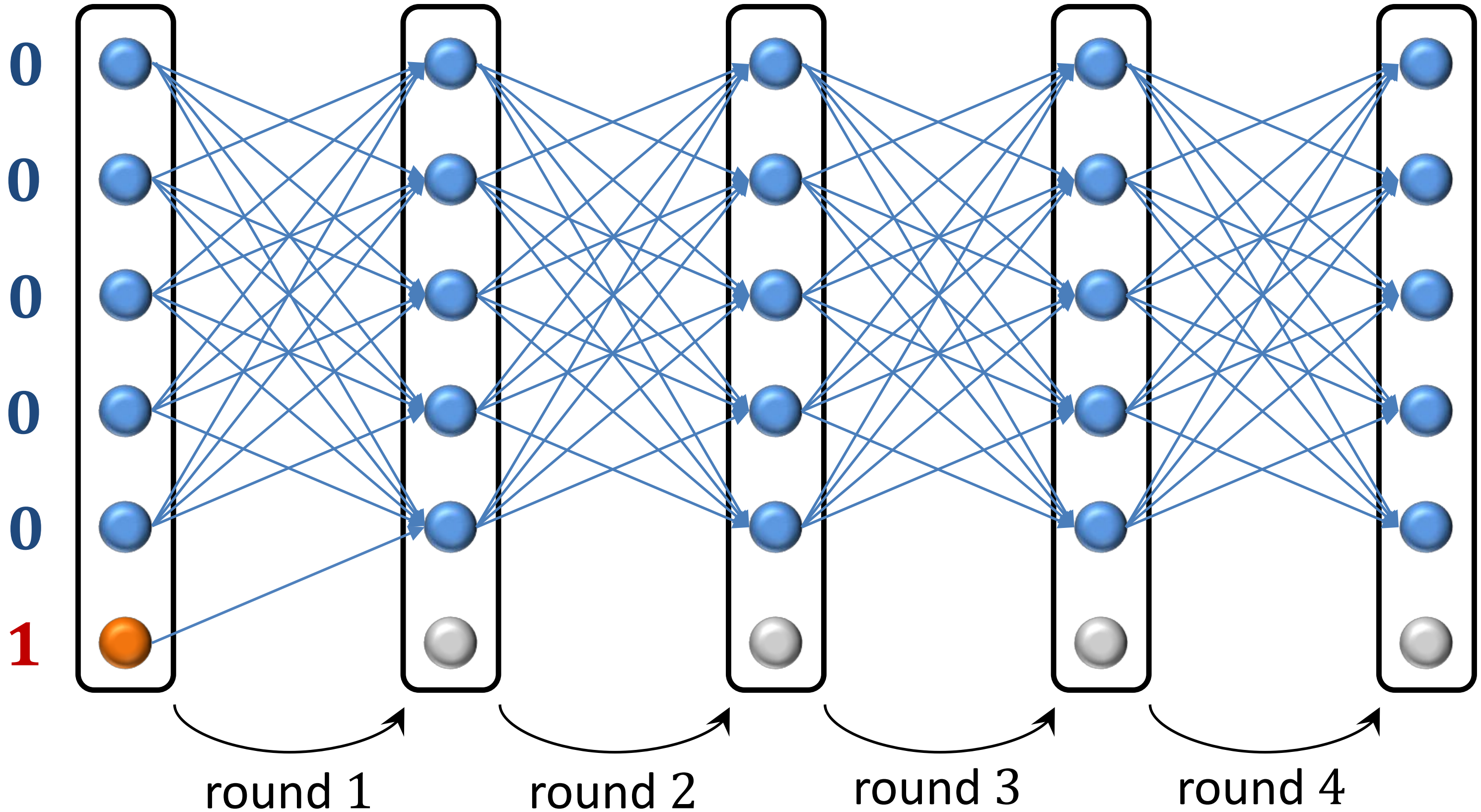

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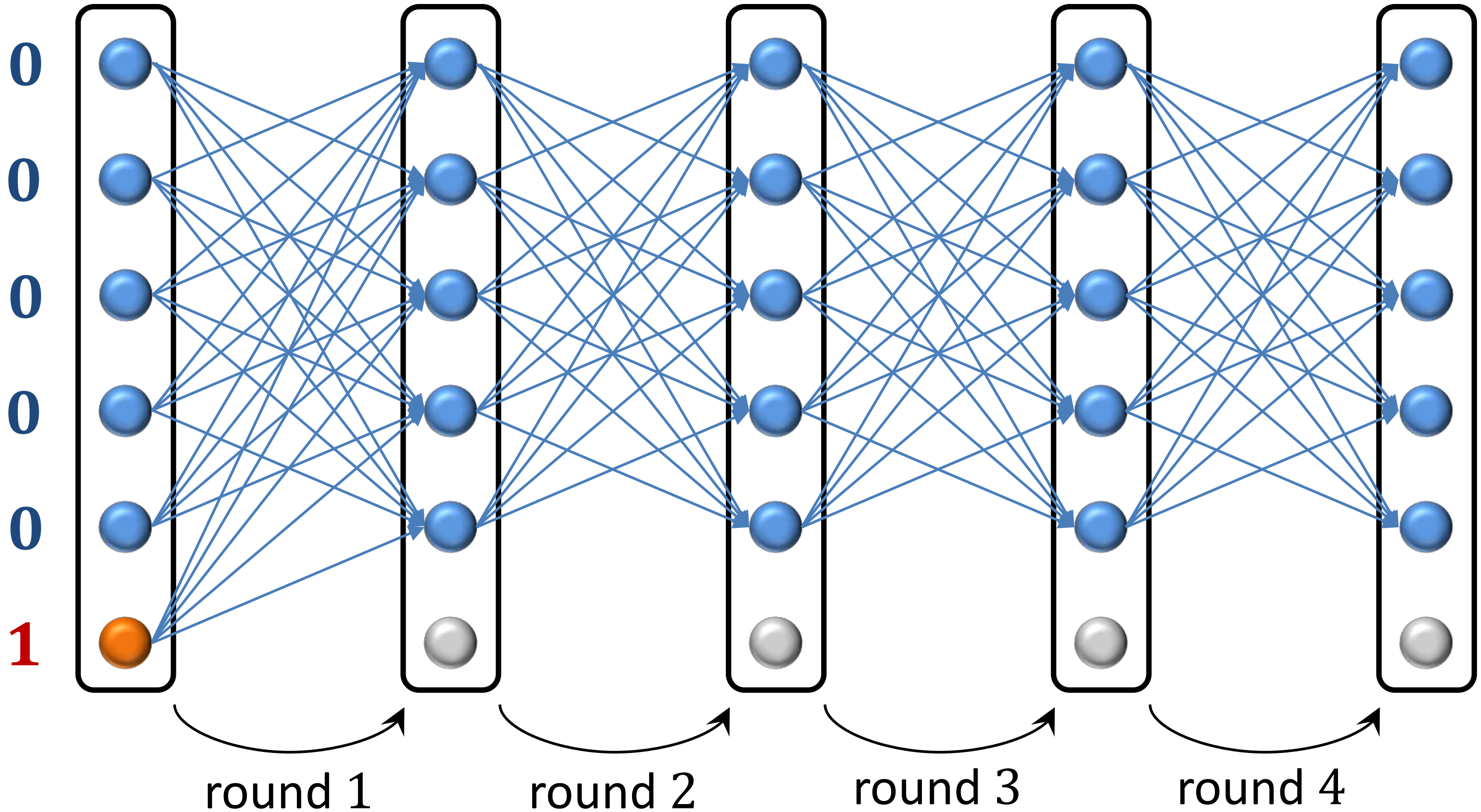

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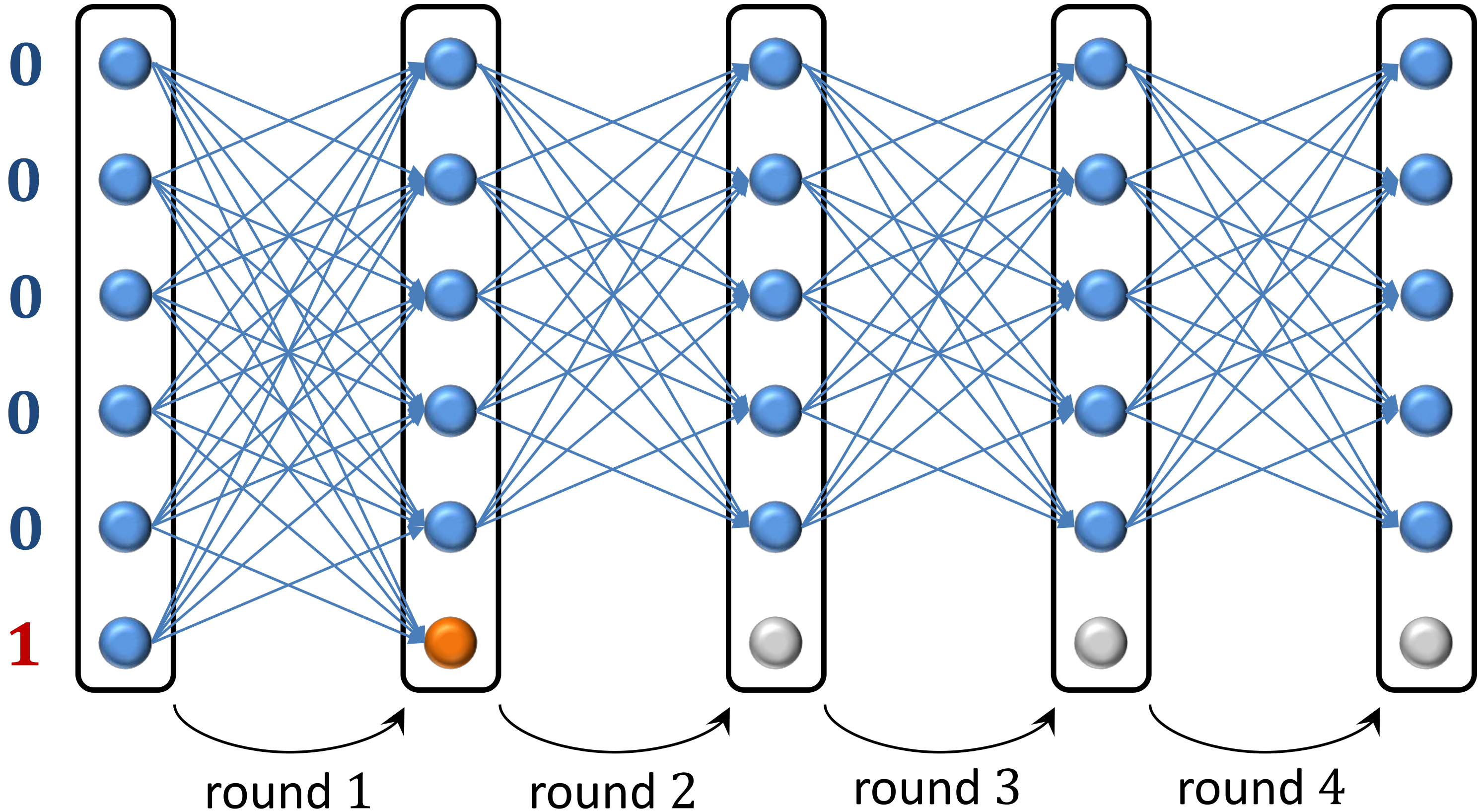

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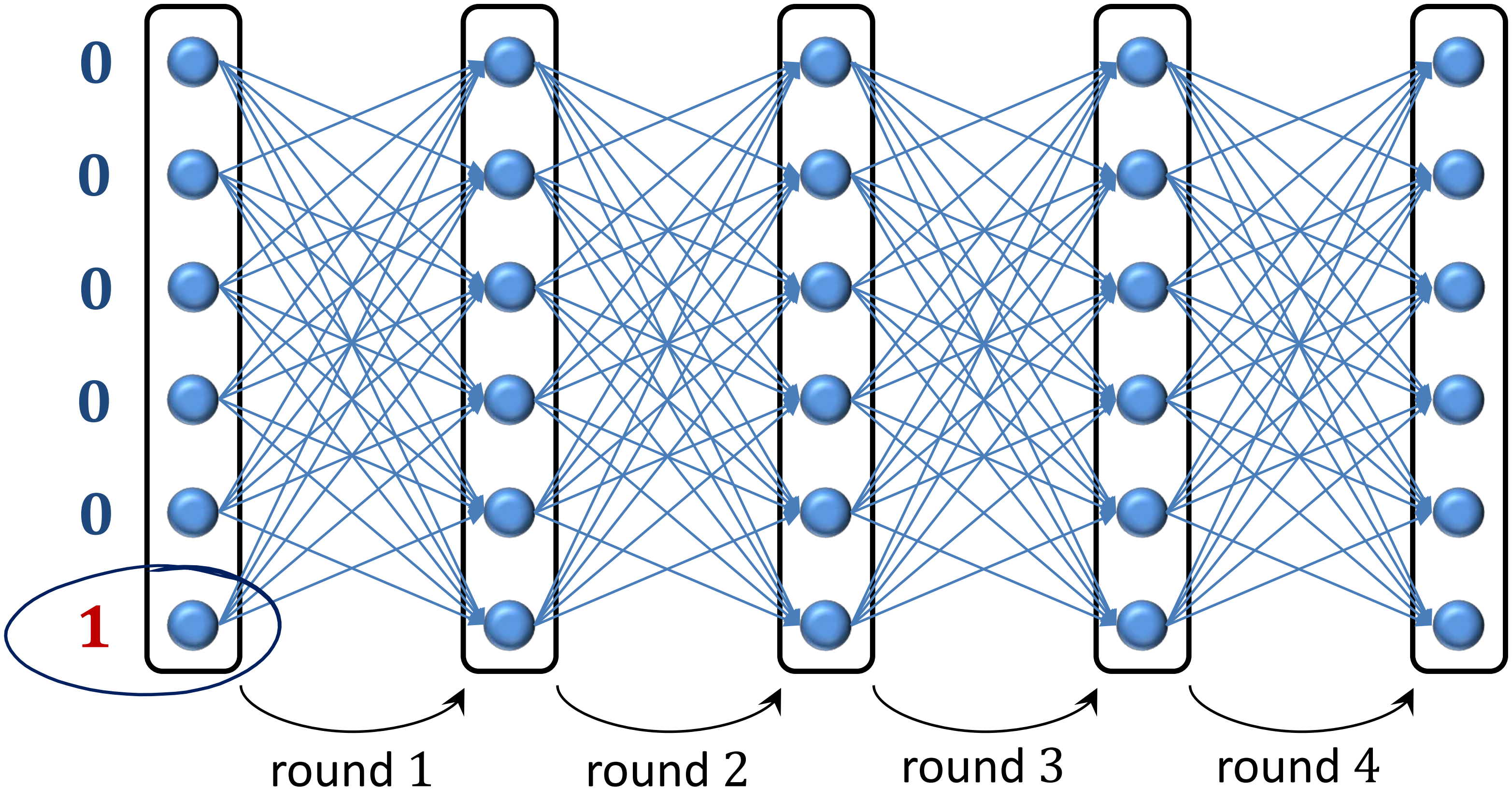

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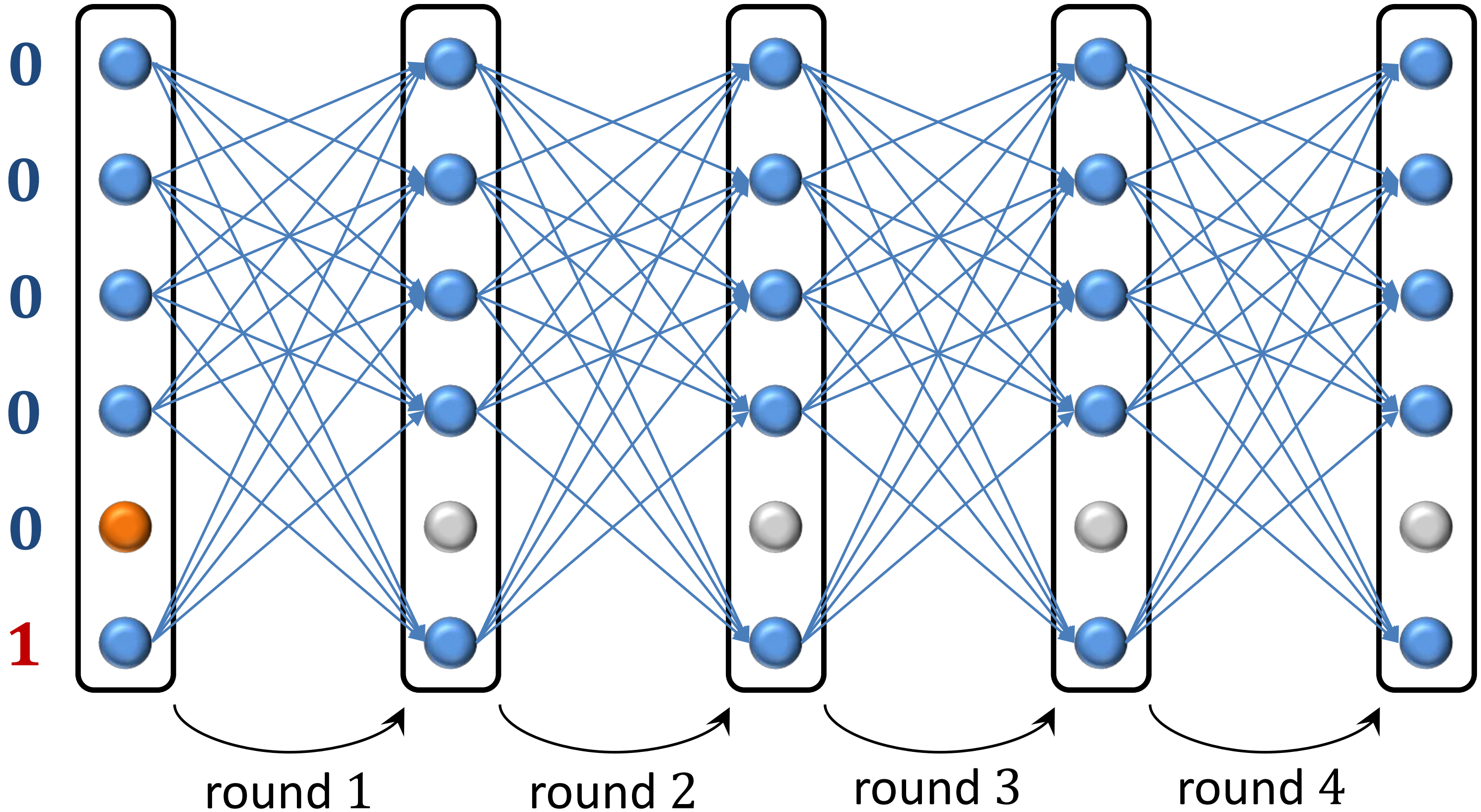

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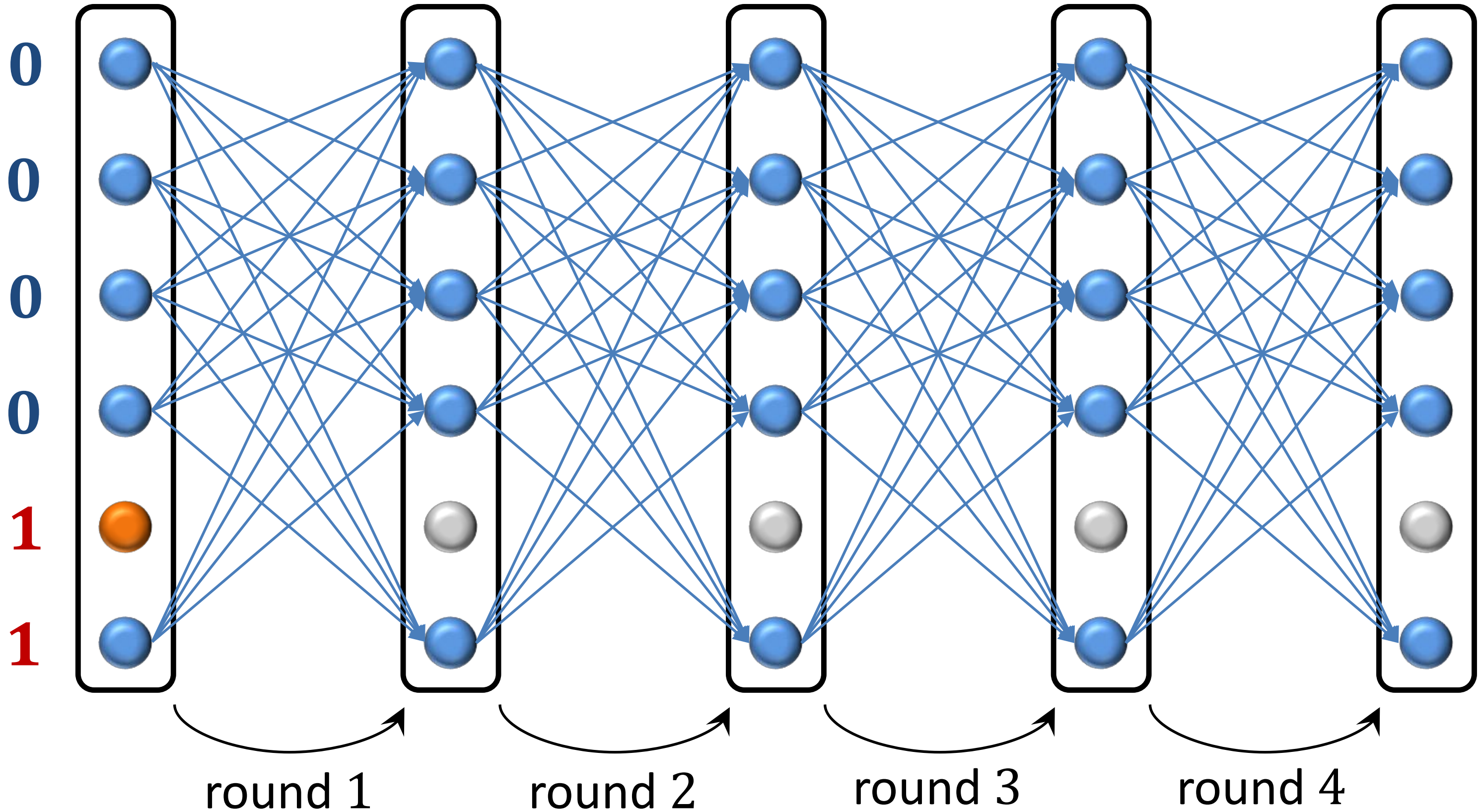

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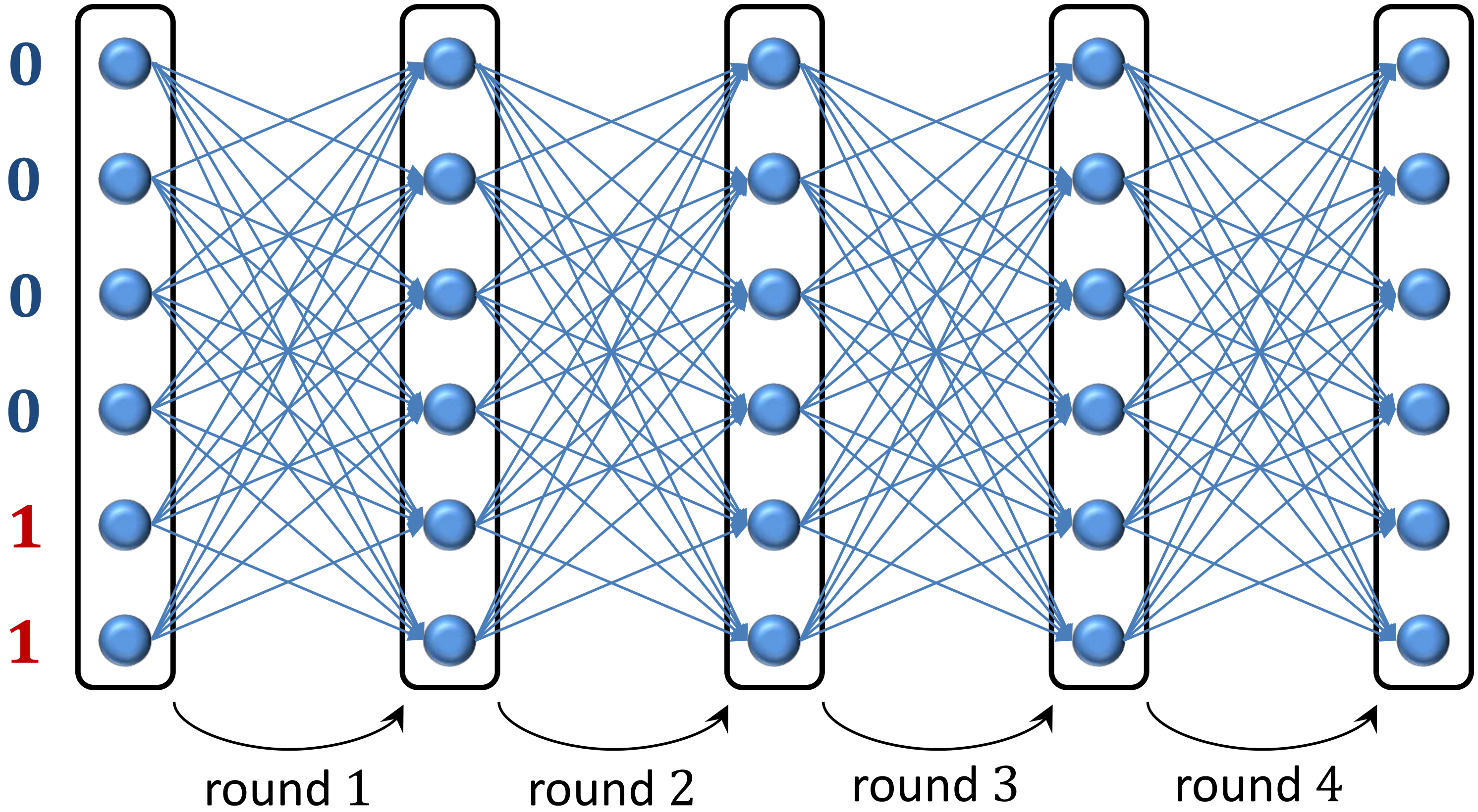

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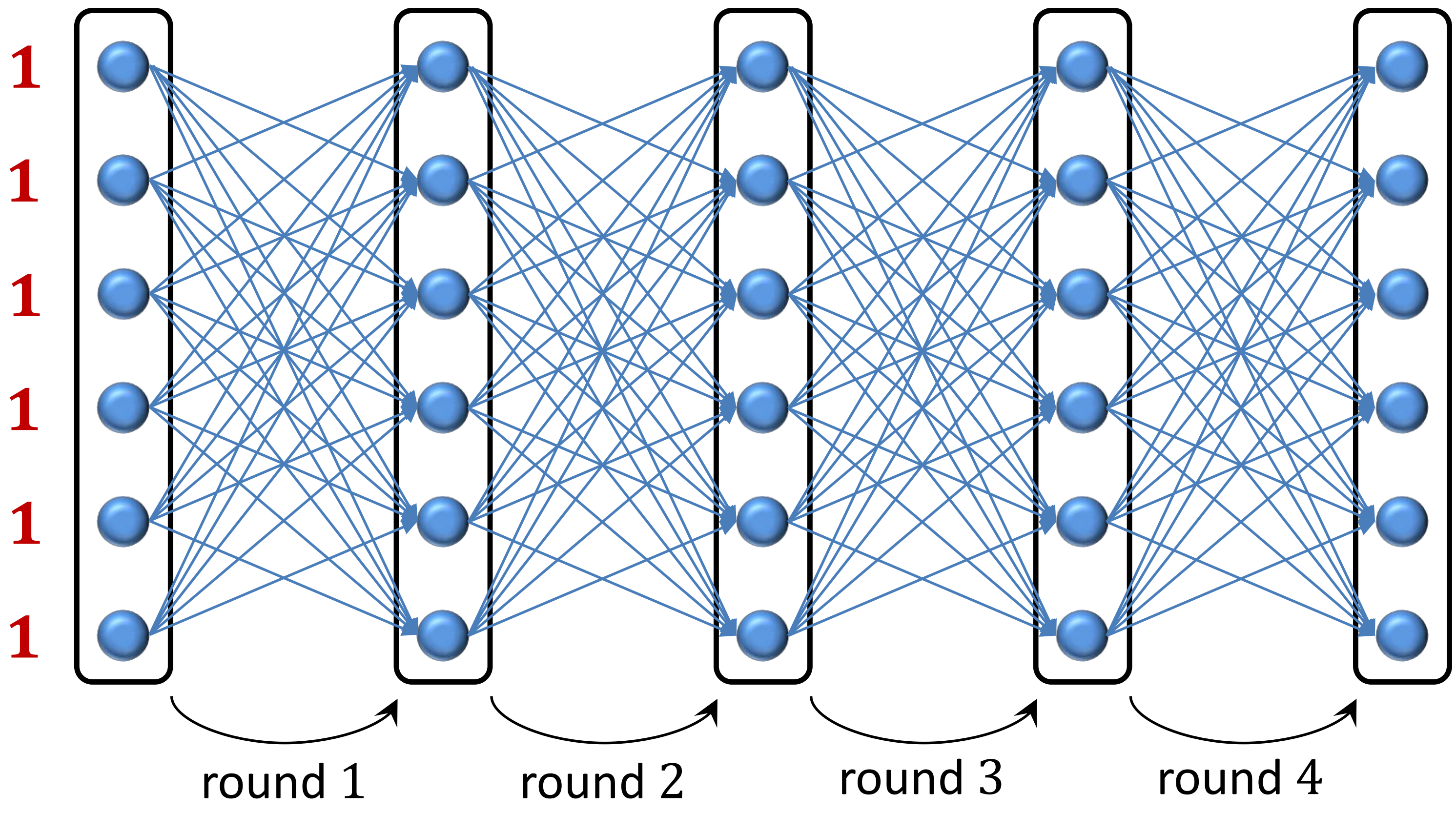


Lower Bound on Rounds

Theorem

If at most $f \leq n - 2$ of n nodes of a synchronous message passing system can crash, at least $f + 1$ rounds are needed to solve consensus.

Proof Sketch:

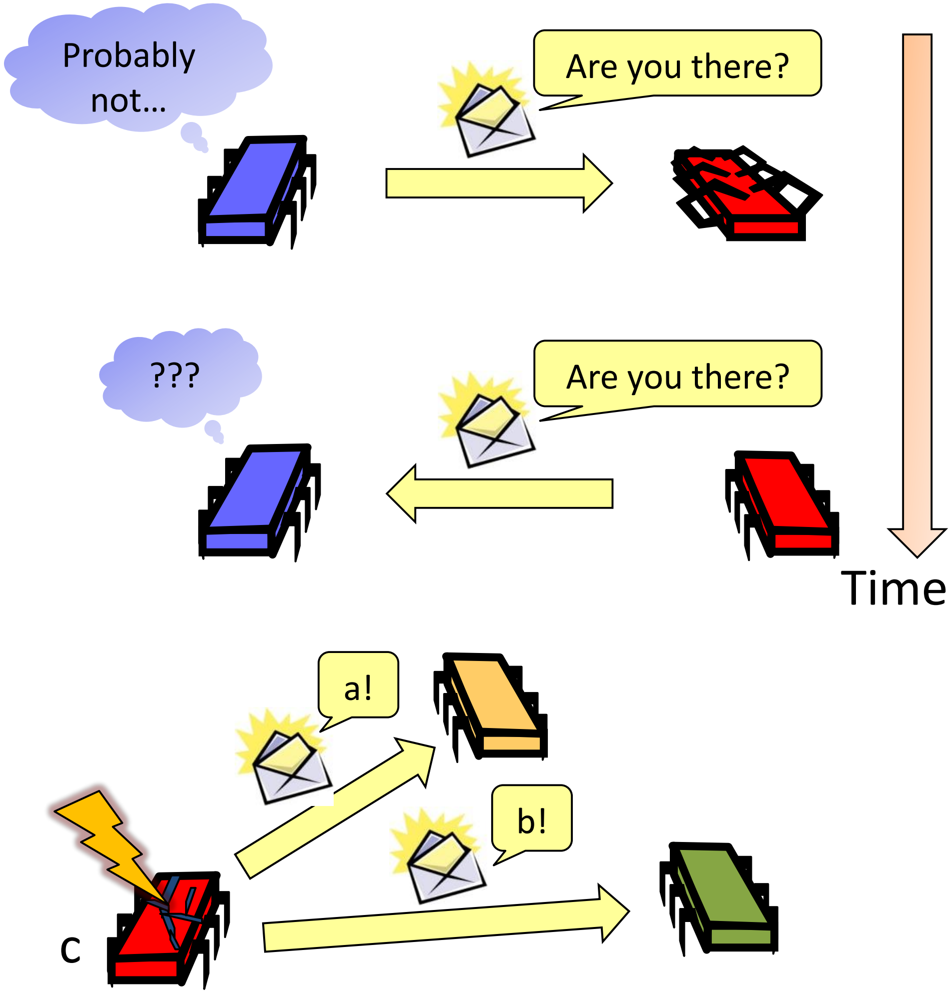
- Similarity chain starting with fault-free all-zeroes execution and ending with fault-free all-ones execution
- In all executions, at most one crash per round
- Construction works as long as there are at least 2 non-faulty nodes in each execution ($n \geq f + 2$)
- **Validity:** all-zeroes $\Longrightarrow$ decision 0; all-ones $\Longrightarrow$ decision 1
  Similarity Chain: same decision in all executions

Arbitrary Behavior

- The assumption that processes crash and stop forever is sometimes too optimistic

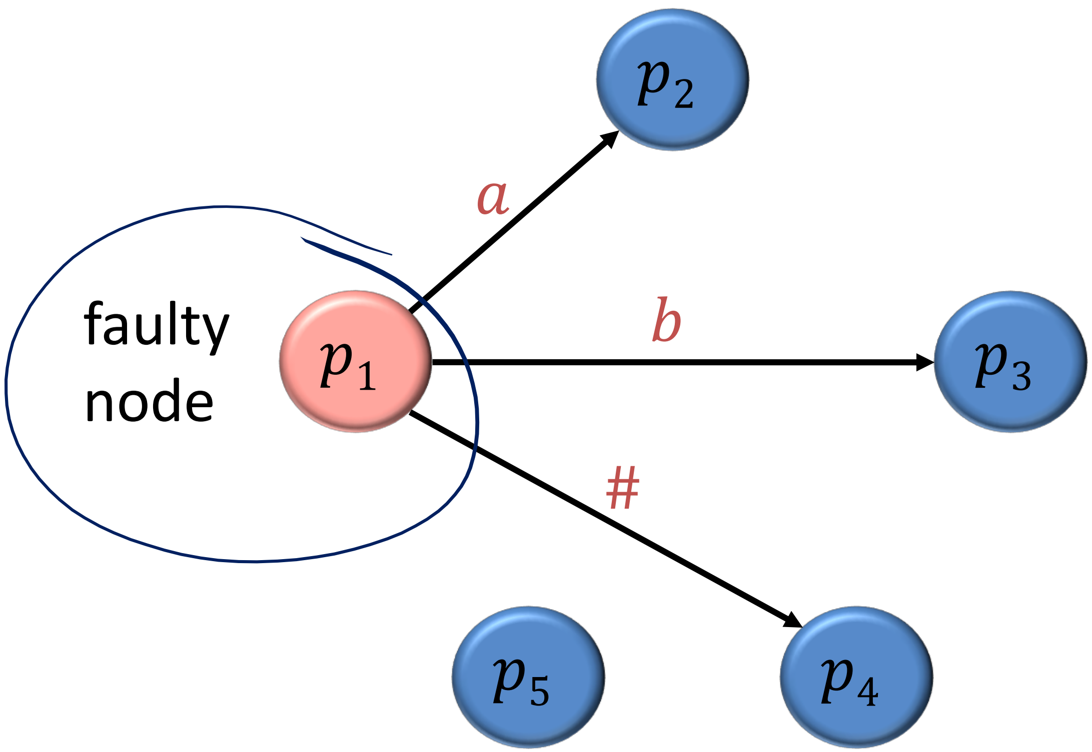
- Maybe the processes fail and recover:

- Maybe the processes are damaged:

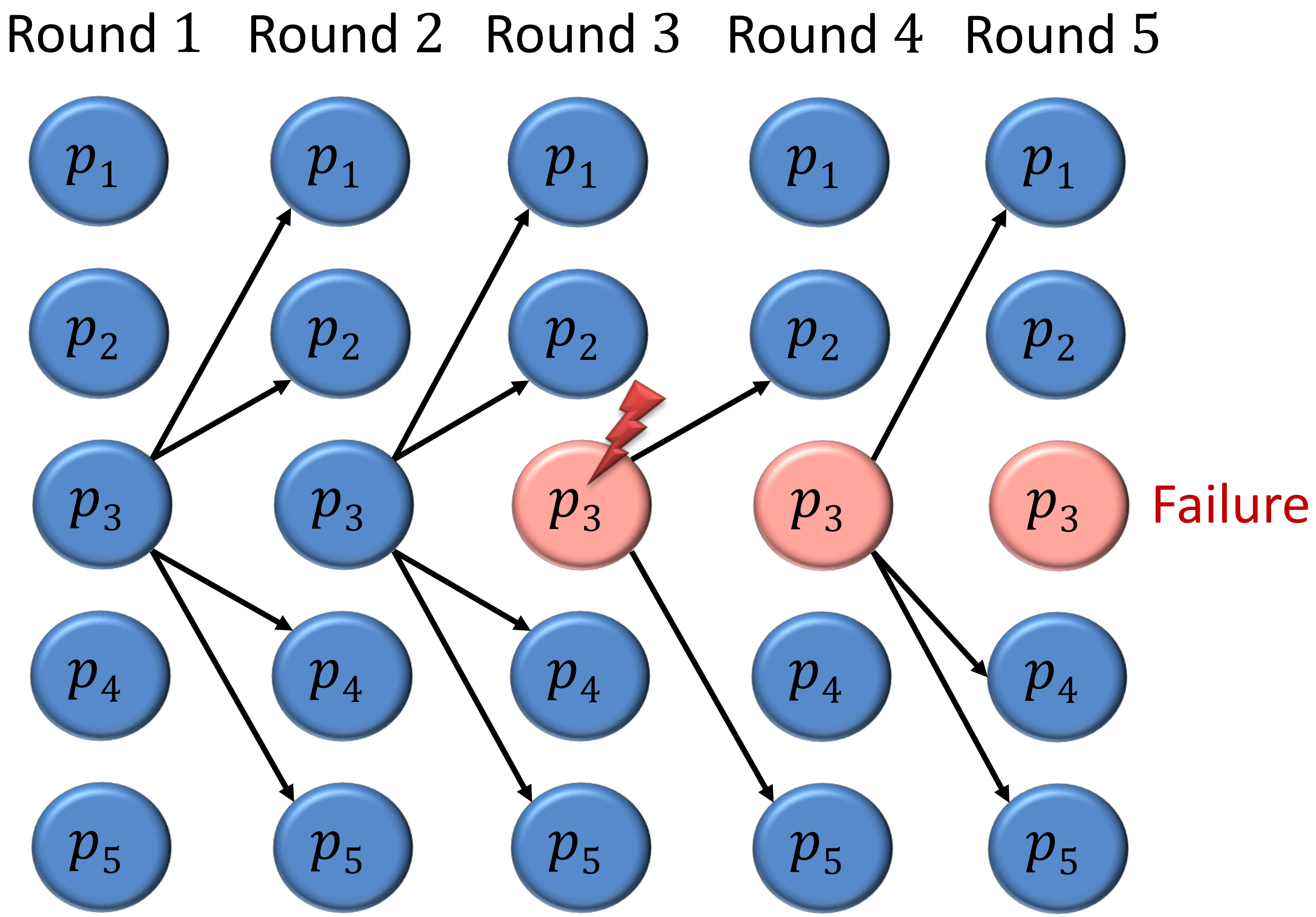


Consensus #5: Byzantine Failures

- Different processes may receive different values
- A Byzantine process can behave like a crash-failed process



After Failure, Node Remains in Network



Consensus with Byzantine Failures

- Again: If an algorithm solves consensus for f failed processes, we say it is an f-resilient consensus algorithm

- **Validity:** If all non-faulty processes start with the same value, then all non-faulty processes decide on that value
  - Note that in general this validity condition does not guarantee that the final value is an input value of a non-Byzantine process
  - However, if the input is binary, then the validity condition ensures that processes decide on a value that at least one non-Byzantine process had initially

- Obviously, any f-resilient consensus algorithm requires at least $f + 1$ rounds (follows from the crash failure lower bound)
- How large can f be…? Can we reach consensus as long as the majority of processes is correct (non-Byzantine)?

Impossibility

Theorem

There is no f-resilient Byzantine consensus algorithm for n nodes for $f \geq n/3$

Proof outline

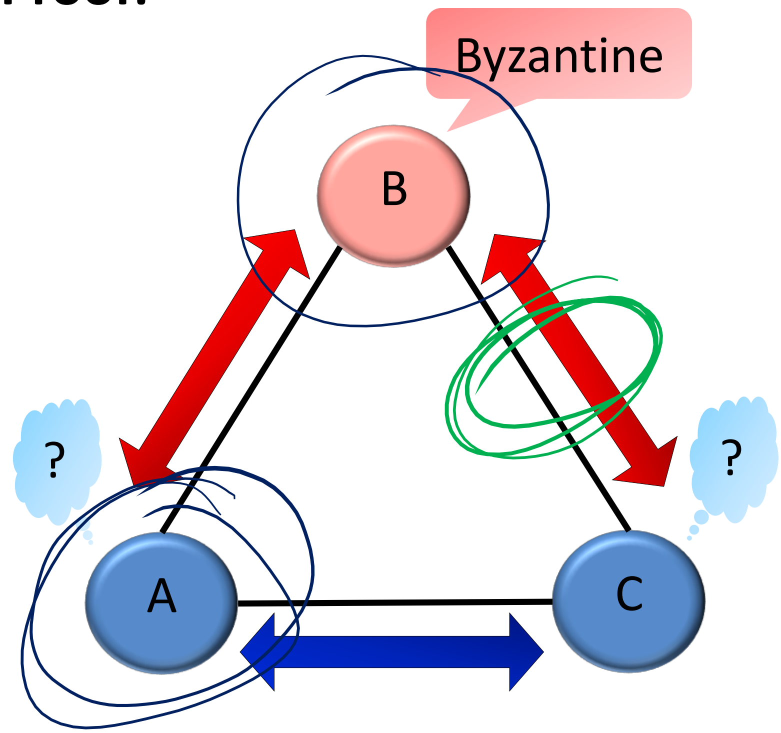
- First, we discuss the 3 node case
  - not possible for $f = 1$
- The general case can then be proved by reduction from the 3 node case
  - Given an algorithm for n node and f faults for $f \geq n/3$, we can construct a 1-resilient 3-node algorithm

The 3 Node Case

Lemma

There is no 1-resilient algorithm for 3 nodes

Proof:

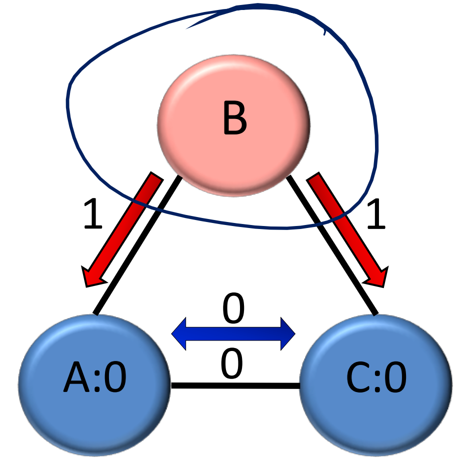


Intuition:

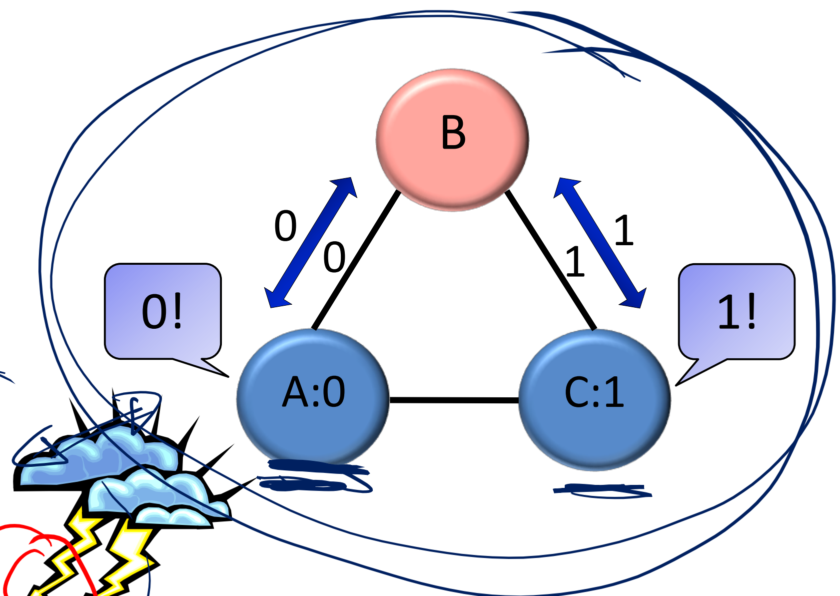
- Node A may also receive information from C about B's messages to C
- Node A may receive conflicting information about B from C and about C from B (the same for C!)
- It is impossible for A and C to decide which information to base their decision on!

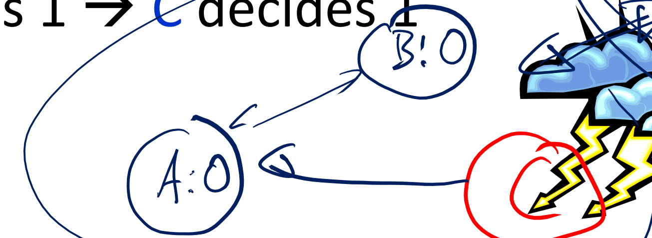
Proof Sketch

- Assume that both A and C have input 0. If they decided 1, they could violate the validity condition → A and C must decide 0 independent of what B says
- Similary, A and C must decide 1 if their inputs are 1
- We see that the processes must base their decision on the majority vote



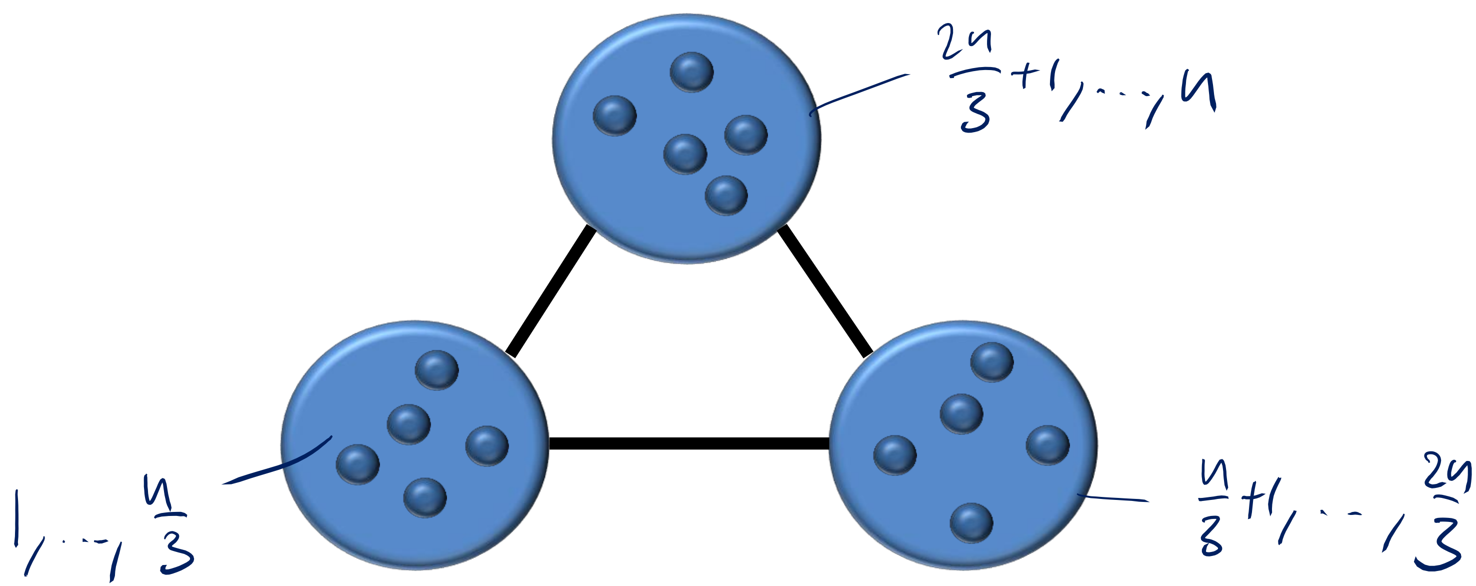
- If A's input is 0 and B tells A that its input is 0 → A decides 0
- If C's input is 1 and B tells C that its input is 1 → C decides 1





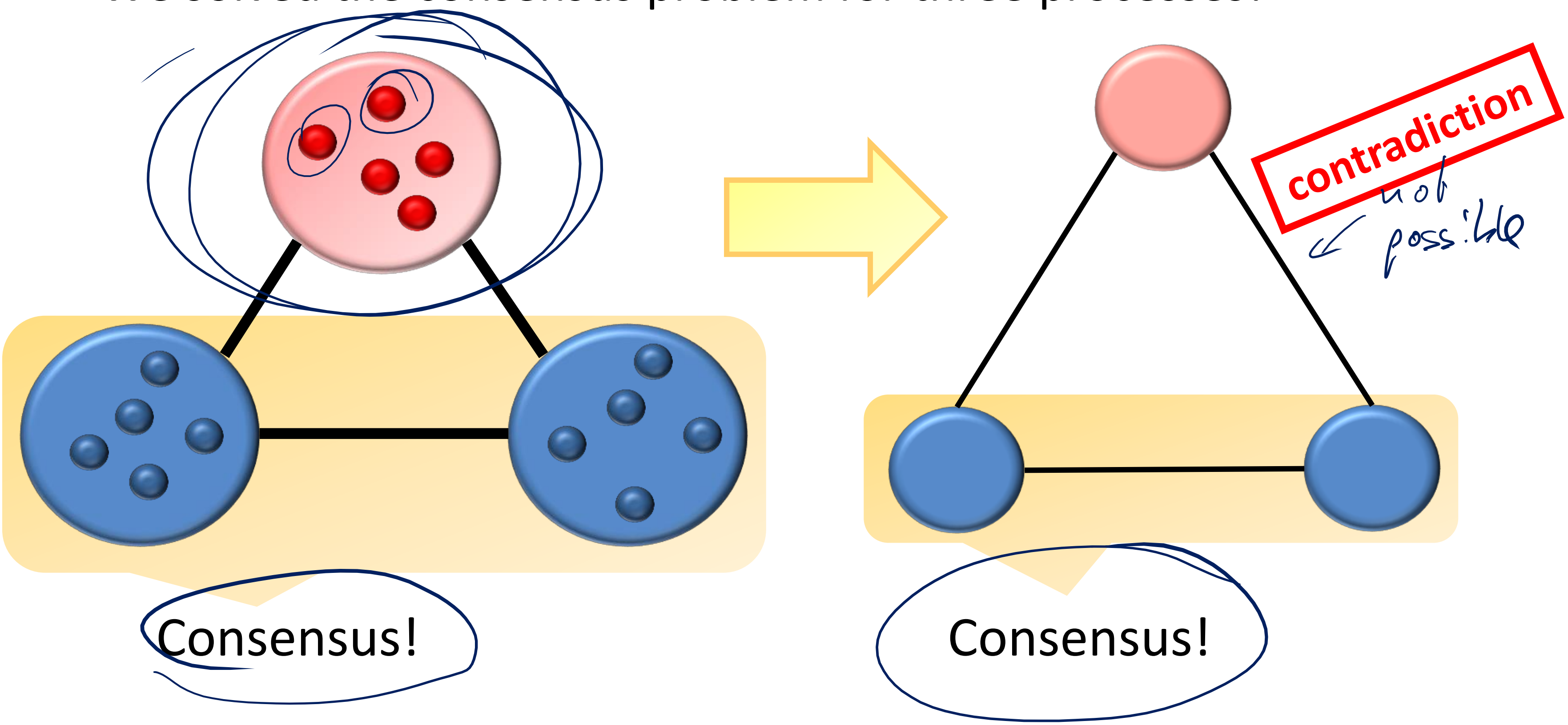
The General Case

- Assume for contradiction that there is an f-resilient algorithm A for n nodes, where $f \geq n/3$
- We use this algorithm to solve consensus for 3 nodes where one node is Byzantine!
- For simplicity assume that n is divisible by 3
- We let each of the three processes simulate $n/3$ processes



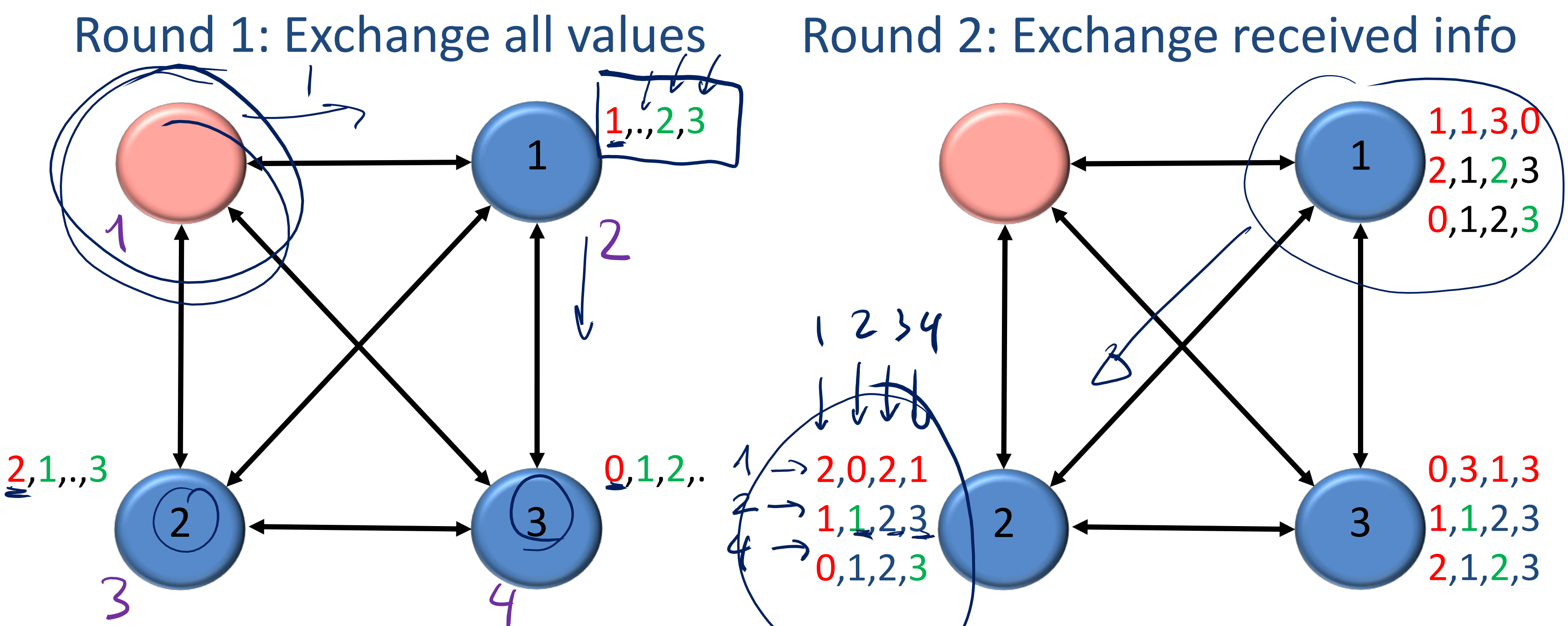
The General Case

- One of the 3 nodes is Byzantine $\Longrightarrow$ its $n/3$ simulated nodes may all behave like Byzantine nodes
- Since algorithm A tolerates $n/3$ Byzantine failures, it can still reach consensus
  $\Longrightarrow$ We solved the consensus problem for three processes!



Cons. #6: Simple Byzantine Agreement Alg.

- Can the nodes reach consensus if $n > 3f$?
- A simpler question: What if $n = 4$ and $f = 1$?
- The answer is yes. It takes two rounds:



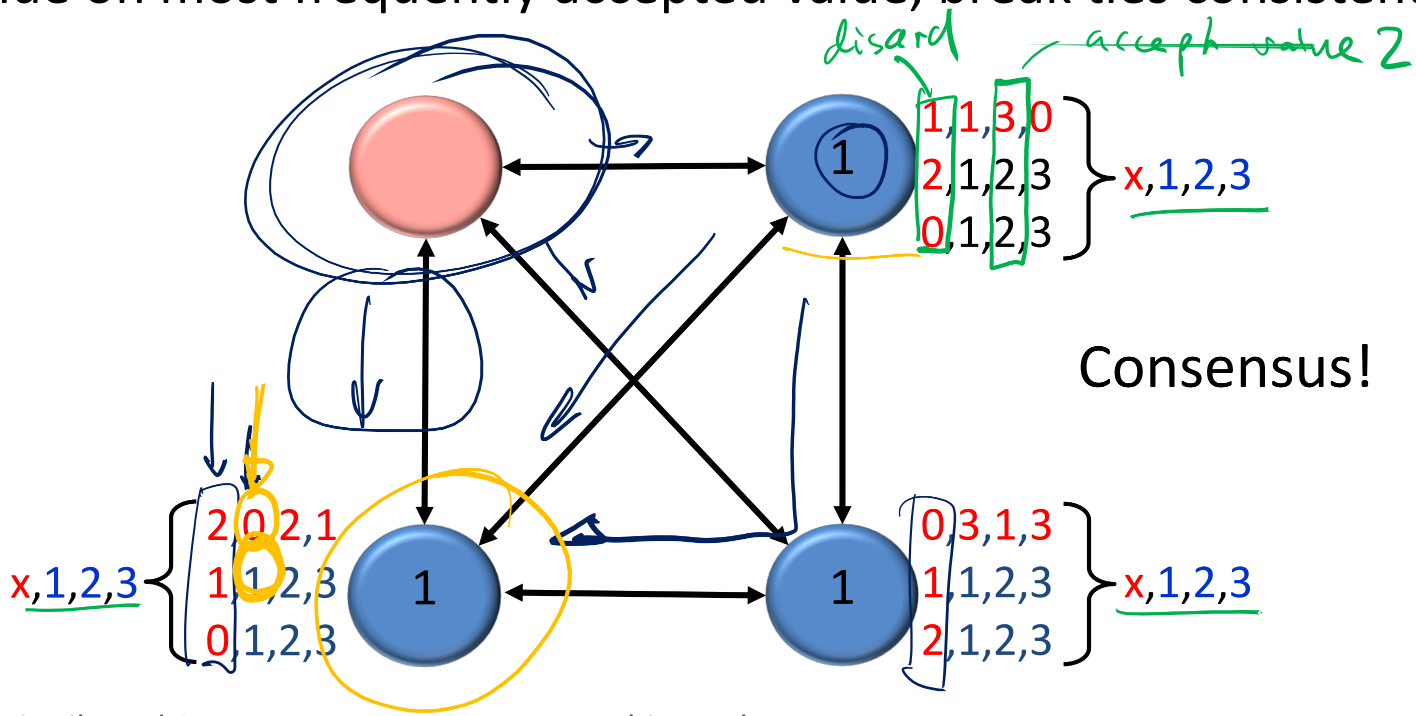
[matrix: one column for each original value, one row for each neighbor]

Simple Byzantine Agreement Algorithm

- After round 2, each node has received 12 values, 3 for each of the 4 input values (columns). If at least 2 of the 3 values of a column are equal, this value is accepted, otherwise it is discarded.
  - Values of honest nodes are accepted
  - The value of the Byzantine node is accepted iff it sends the same value to at least two nodes in the first round.

← pick min.

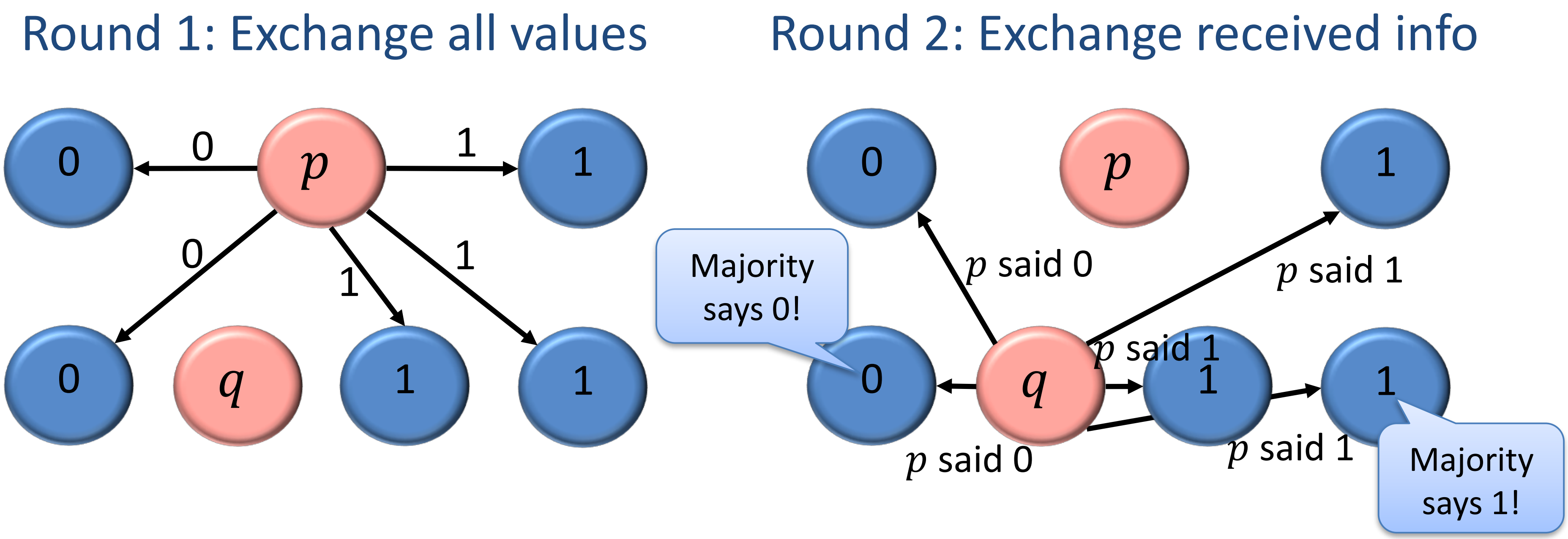
- Decide on most frequently accepted value, break ties consistently!



Consensus!

Simple Byzantine Agreement Algorithm

- Does the algorithm still work in general for any f and $n > 3f$?
- The answer is no. Try $f = 2$ and $n = 7$:



- The problem is that q can say different things about what p sent to q
  - It can also obviously not work because of the $f + 1$ round lower bound.
  - What is the solution to this problem?

Simple Byzantine Agreement Algorithm

- The solution is simple: Again exchange all information!
- This way, the nodes can learn that q gave inconsistent information about p
- Hence, q can be excluded, and also p if it also gave inconsistent information (about q).
- If $f = 2$ and $n > 6$, consensus can be reached in 3 rounds!
- In fact, the following "algorithm" solves the problem for any f and any $n > 3f$:

> Exchange all information for $f + 1$ rounds
> Ignore all nodes that provided inconsistent information
> Let all nodes decide based on the same input

Simple Byzantine Agreement Algorithm

The proposed algorithm has several advantages:

- \+ It works for any f and $n > 3f$, which is optimal
- \+ It only takes $f + 1$ rounds. This is even optimal for crash failures!
- \+ It works for any input and not just binary input

However, it has some considerable disadvantages:

- – "Ignoring all nodes that provided inconsistent information" is not easy to formalize
- – The **size of the messages increases exponentially**! This is a severe problem. It is therefore worth studying whether it is possible to solve the problem with small(er) messages

Consensus #7: The Queen Algorithm

- The Queen algorithm is a simple Byzantine agreement algorithm that uses small messages
- The Queen algorithm solves consensus with n nodes and f failures where $f < n/4$ in $f + 1$ phases

A phase consists of 2 rounds

Idea:

- There is a different (a priori known) queen in each phase
- Since there are $f + 1$ phases, in one phase the queen is not Byzantine
- Make sure that in this round all nodes choose the same value and that in future rounds the nodes do not change their values anymore

The Queen Algorithm

In each phase $i \in \{1, \ldots, f+1\}$:

At the end of phase $f+1$, decide on own value

Round 1:

Broadcast own value

Also send own value to oneself

Set own value to the value that was received most often

If own value appears $> n/2 + f$ times

support this value

else

do not support any value

$f < n/4$

If several values have the same (highest) frequency, choose any value, e.g., the smallest

Round 2:

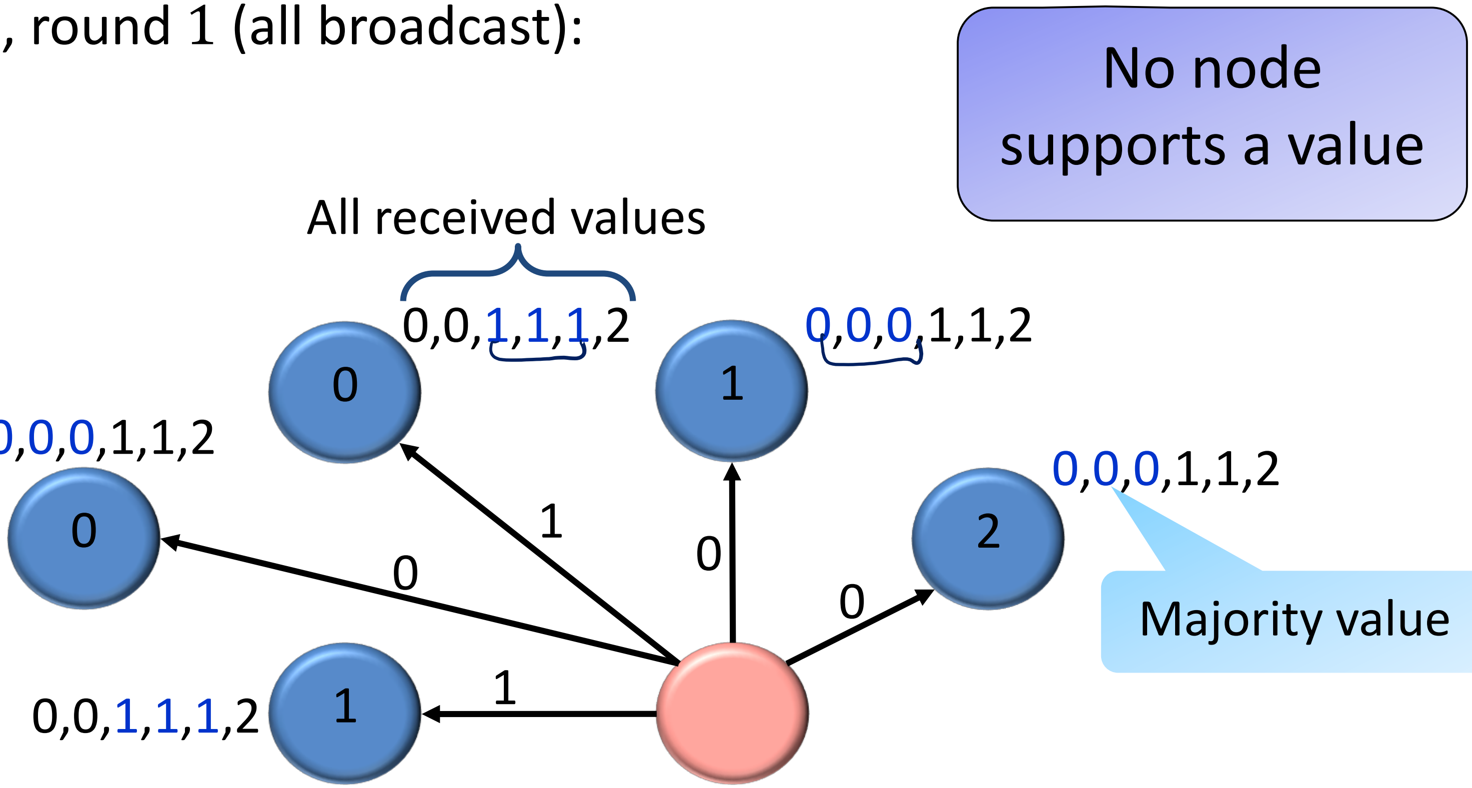
The queen broadcasts its value

If not supporting any value

set own value to the queen's value

The Queen Algorithm: Example

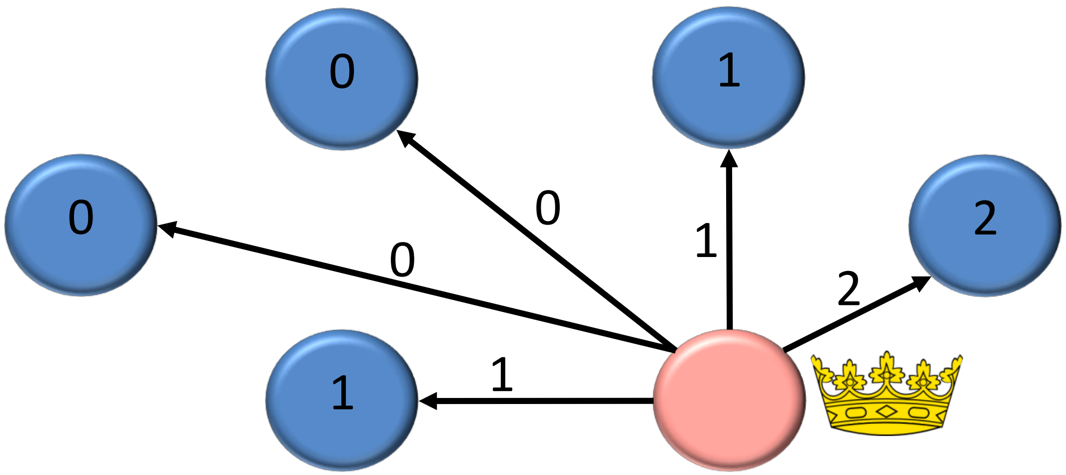
- Example: $n = 6, f = 1$
- Phase 1, round 1 (all broadcast):



The Queen Algorithm: Example

- Example: $n = 6, f = 1$
- Phase 1, round 2 (queen broadcasts):

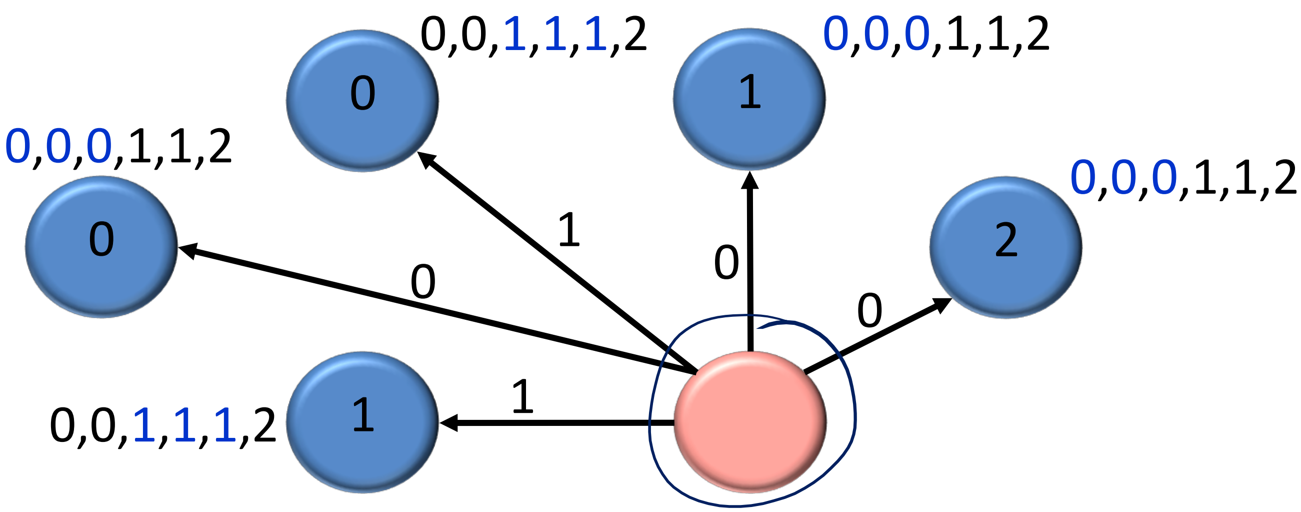
All nodes choose the queen's value



The Queen Algorithm: Example

- Example: $n = 6, f = 1$
- Phase 2, round 1 (all broadcast):

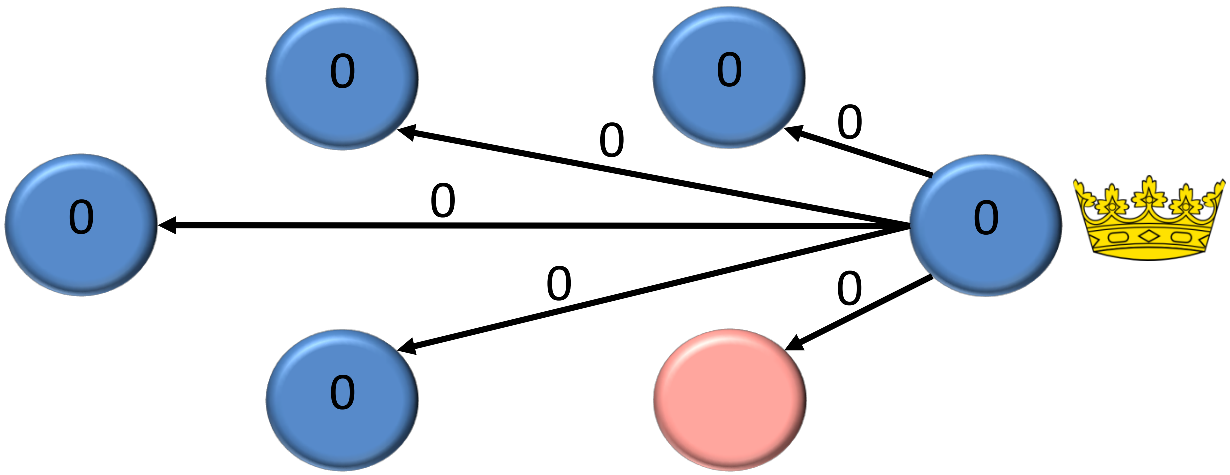
No node supports a value



The Queen Algorithm: Example

- Example: $n = 6, f = 1$
- Phase 2, round 2 (queen broadcasts):

All nodes choose the queen's value



Consensus!

The Queen Algorithm: Analysis

- After the phase where the queen is correct, all correct nodes have the same value
  - If all nodes change their values to the queen's value, obviously all values are the same
  - If some node does not change its value to the queen's value, it received a value $> n/2 + f$ times → All other correct nodes (including the queen) received this value $> n/2$ times and thus all correct nodes share this value
- In all future phases, no node changes its value
  - In the first round of such a phase, nodes receive their own value from at least $n - f > n/2$ nodes and thus do not change it
  - The nodes do not accept the queen's proposal if it differs from their own value in the second round because the nodes received their own value at least $n - f > n/2 + f$ times. Thus, all correct nodes support the same value

That's why we need $f < n/4$!

The Queen Algorithm: Summary

The Queen algorithm has several advantages:

+ The messages are small: nodes only exchange their current values
+ It works for any input and not just binary input

However, it also has some disadvantages:

– The algorithm requires $f + 1$ phases consisting of 2 rounds each ... this is twice as much as an optimal algorithm

– It only works with $f < n/4$ Byzantine nodes!

- Is it possible to get an algorithm that works with $f < n/3$ Byzantine nodes and uses small messages?

Consensus #8: The King Algorithm

- The King algorithm is an algorithm that tolerates $f < n/3$ Byzantine failures and uses small messages
- The King algorithm also takes $f + 1$ phases

A phase now consists of 3 rounds

Idea:

- The basic idea is the same as in the Queen algorithm
- There is a different (a priori known) king in each phase
- Since there are $f + 1$ phases, in one phase the king is not Byzantine
- The difference to the Queen algorithm is that the correct nodes only propose a value if many nodes have this value, and a value is only accepted if many nodes propose this value

The King Algorithm

In each phase $\boldsymbol{i} \in \{\mathbf{1} \ldots \boldsymbol{f} + \mathbf{1}\}$:

At the end of phase $f + 1$, decide on own value

Round 1:

Broadcast own value

Also send own value to oneself

Round 2:

If some value x appears $\geq n - f$ times

Broadcast "Propose x"

If some proposal received $> f$ times

Set own value to this proposal

Round 3:

The king broadcasts its value

If own value received $< n - f$ proposals

Set own value to the king's value

The King Algorithm: Summary

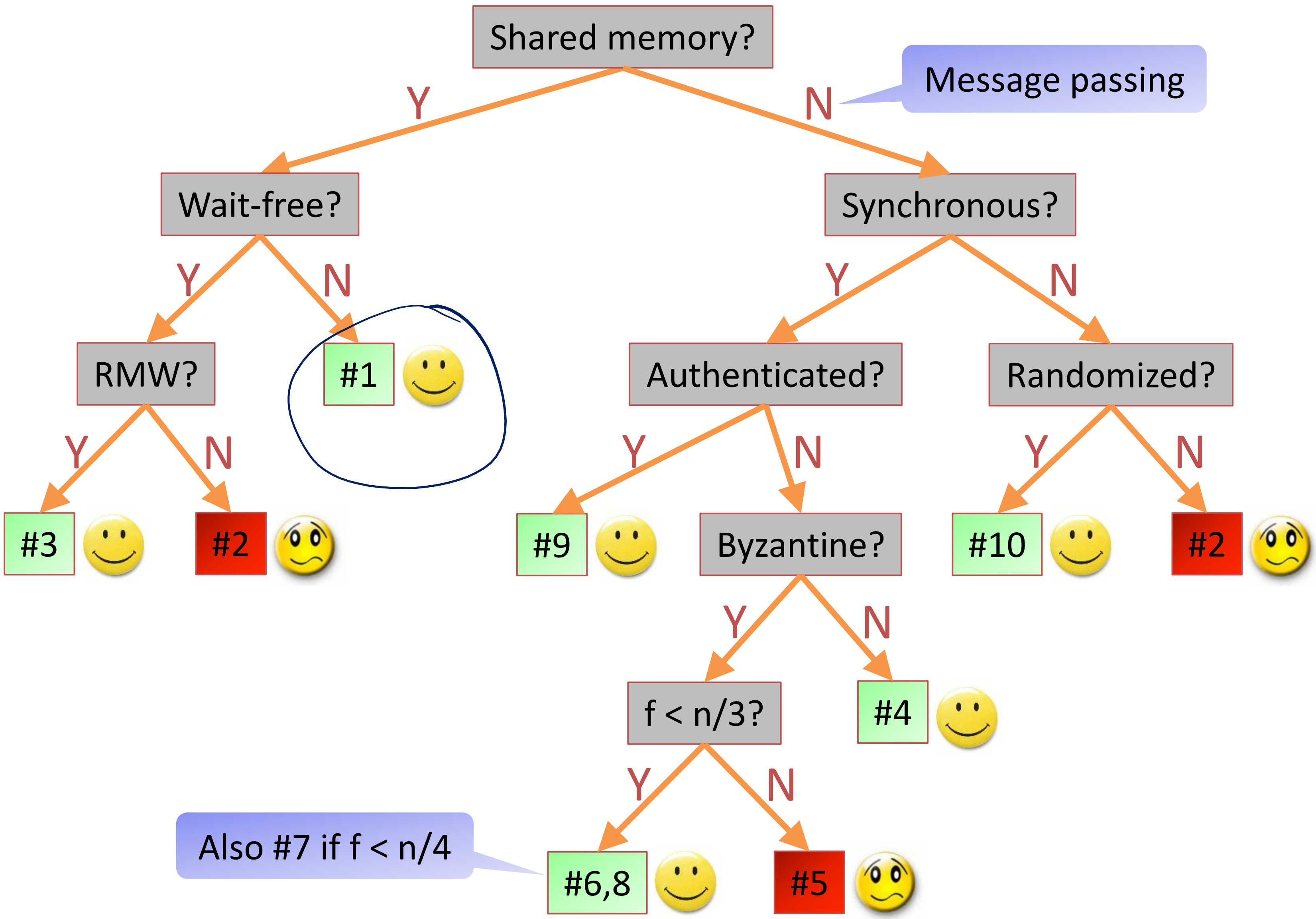
The King algorithm has several advantages:

+ It works for any f and $n > 3f$, which is optimal
+ The messages are small: processes only exchange their current values
+ It works for any input and not just binary input

However, it also has a disadvantage:

– The algorithm requires $f + 1$ phases consisting of 3 rounds each
  This is three times as much as an optimal algorithm

Consensus: Decision Tree



Credits

- The impossibility result (#2) is from Fischer, Lynch, Paterson, 1985
- The hierarchy (#3) is from Herlihy, 1991.
- The synchronous studies (#4) are from Dolev and Strong, 1983, and others.
- The Byzantine agreement problem (#5) and the simple algorithm (#6) are from Lamport, Shostak, Pease, 1980ff., and others
- The Queen algorithm (#7) and the King algorithm (#8) are from Berman, Garay, and Perry, 1989.
- The algorithm using authentication (#9) is due to Dolev and Strong, 1982.
- The first randomized algorithm (#10) is from Ben-Or, 1983.
- The concept of a shared coin was introduced by Bracha, 1984.
- Byzantine Agreement with fewer than n^2 messages is from King and Saia 2011.
- Byzantine Agreement in an asynchronous setting is from King and Saia 2013.