Lower Bounds

Dennis Olivetti

University of Freiburg, Germany

Lower Bounds

"*The 2-coloring problem requires* **$\Omega(n)$** rounds"

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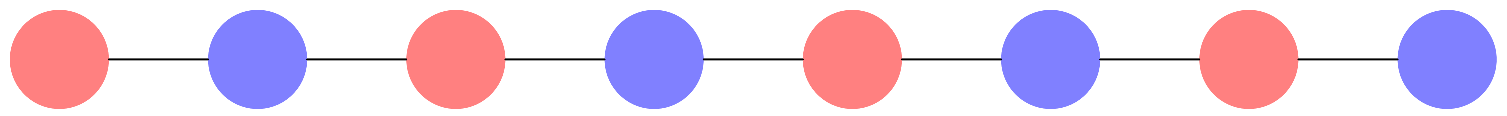
- **2** coloring



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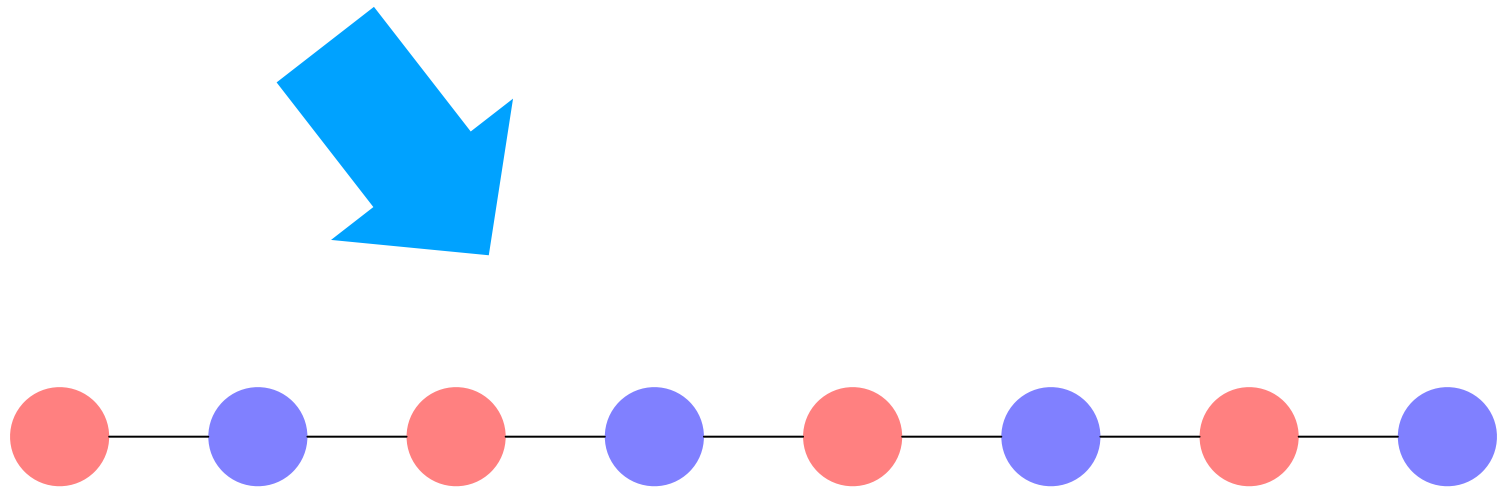

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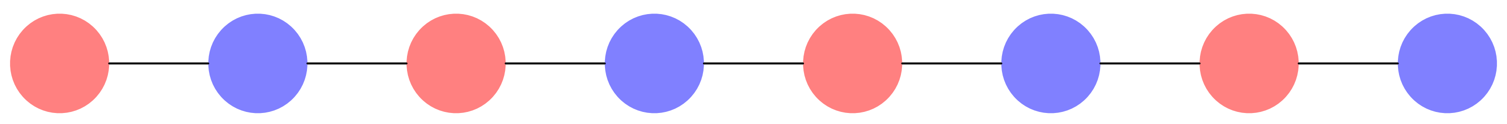

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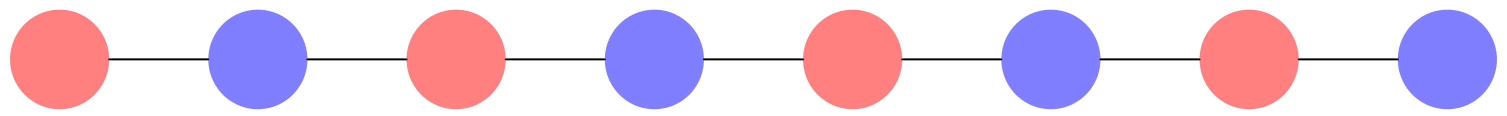

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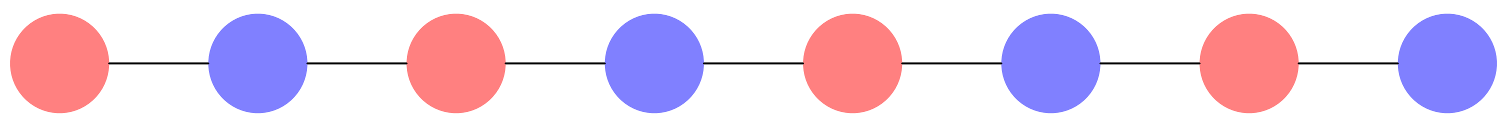

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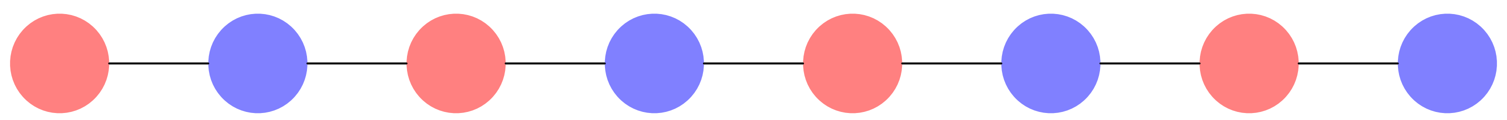

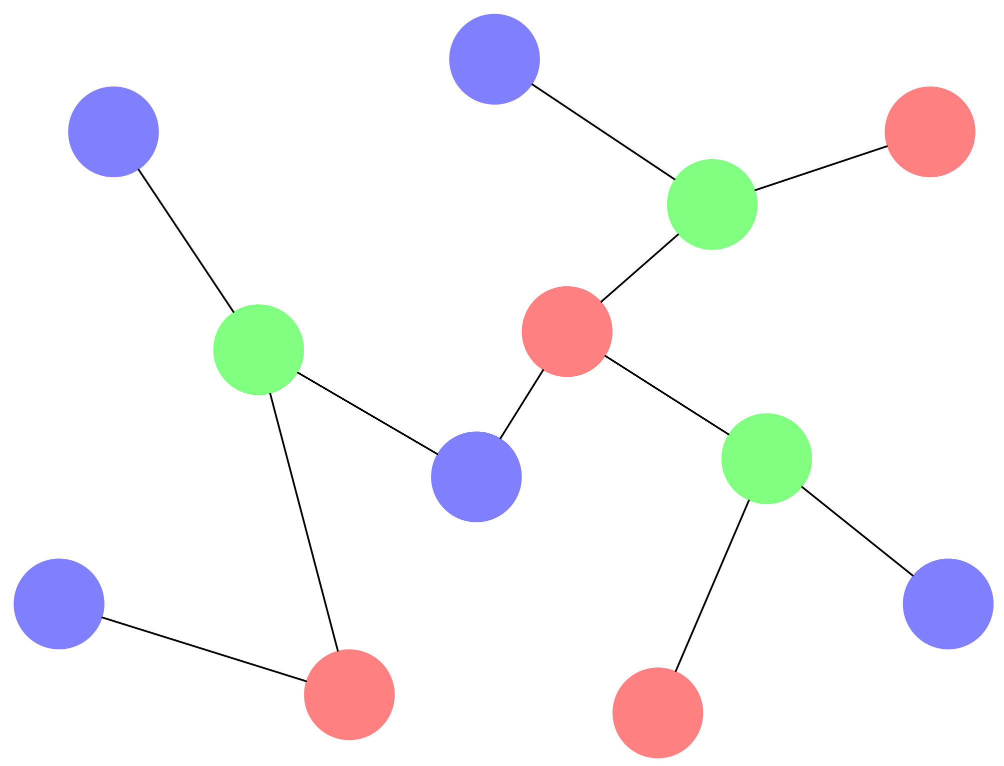

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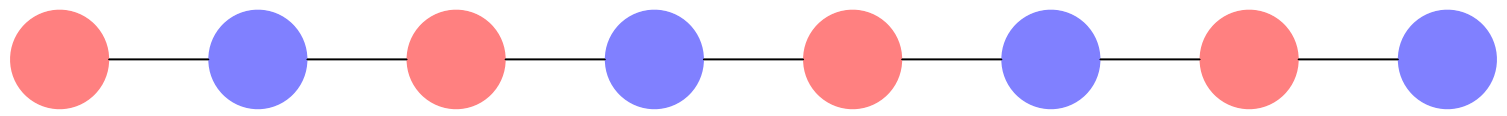

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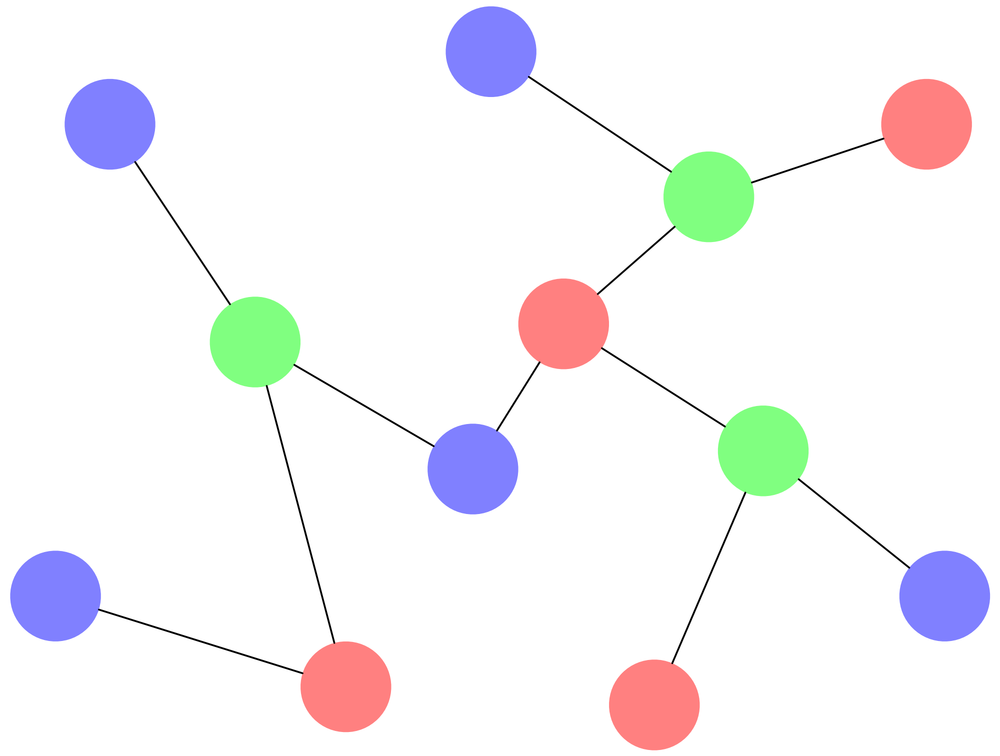

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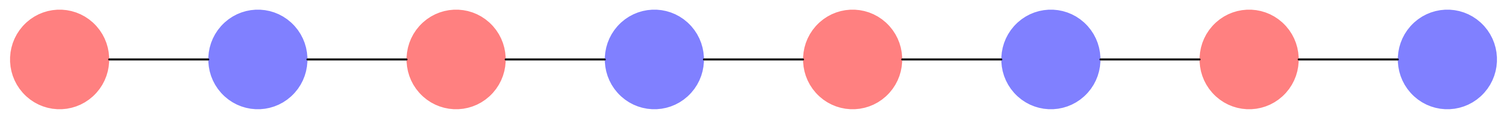

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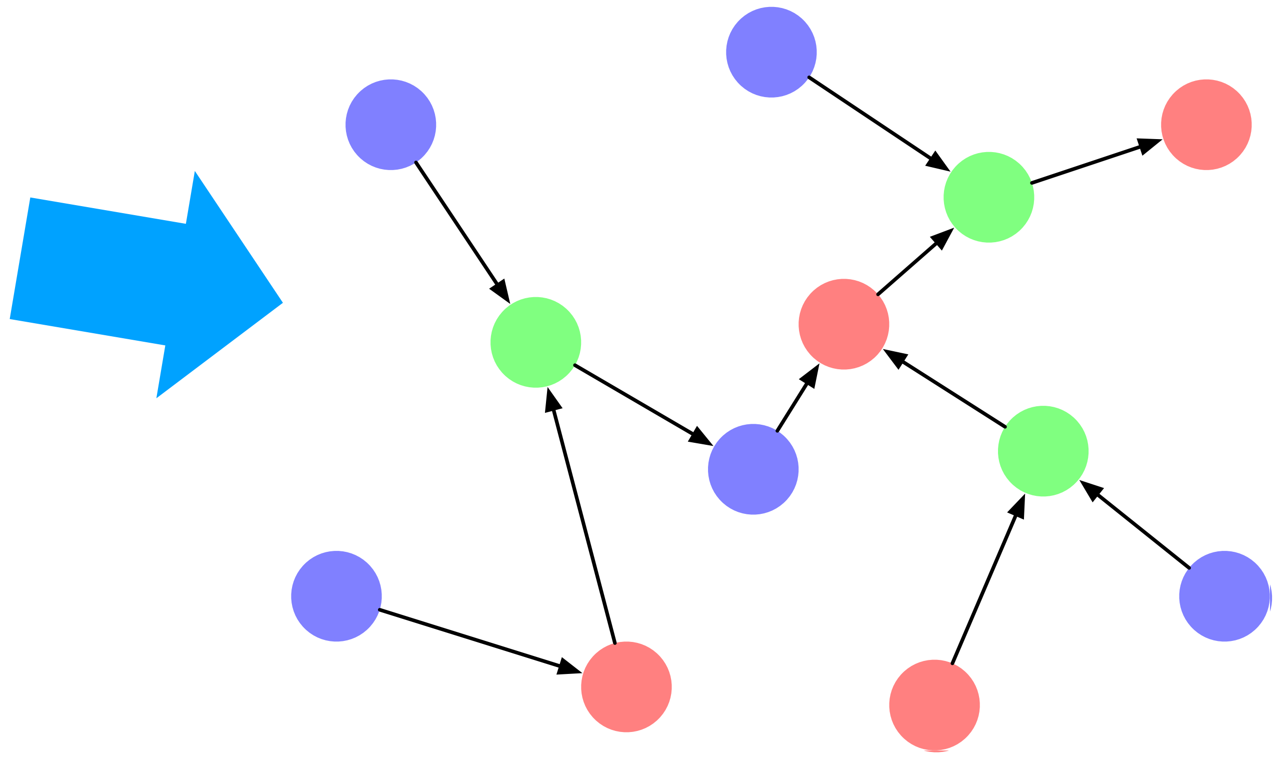

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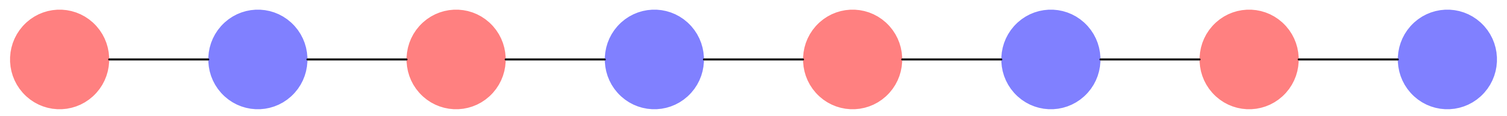

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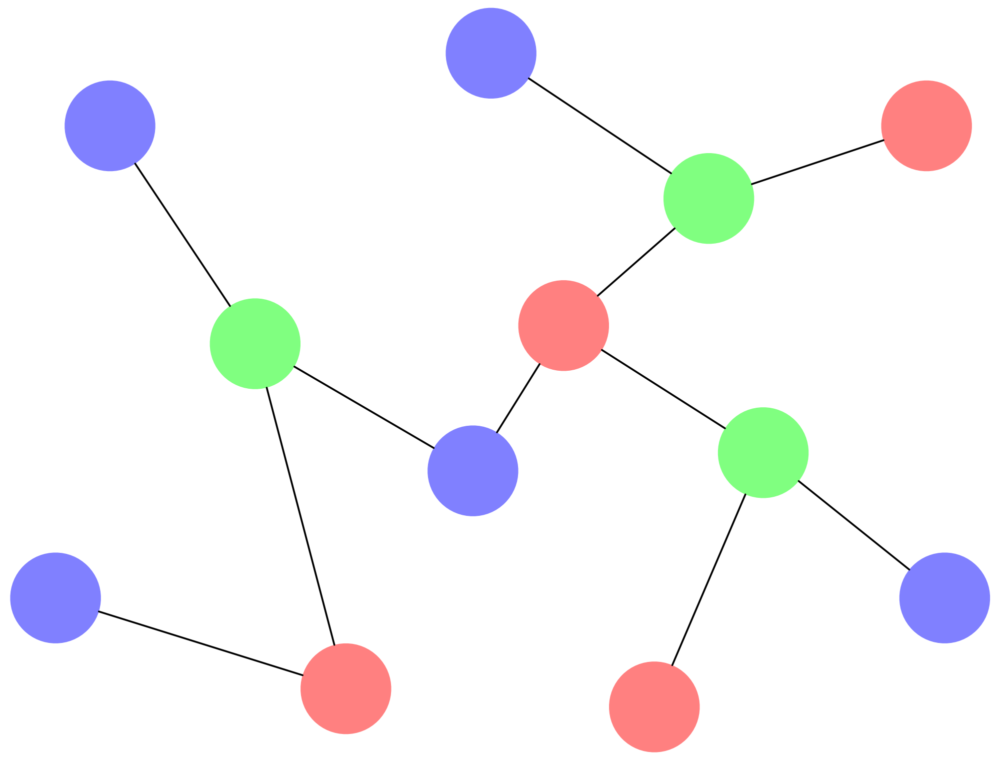

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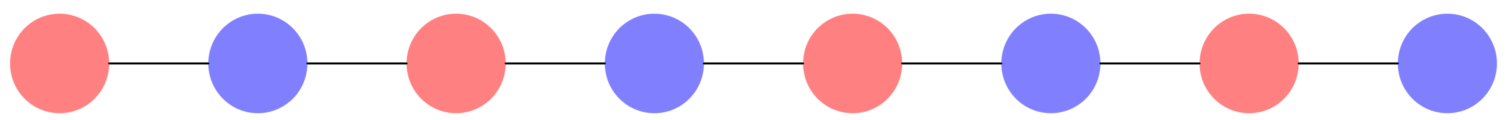

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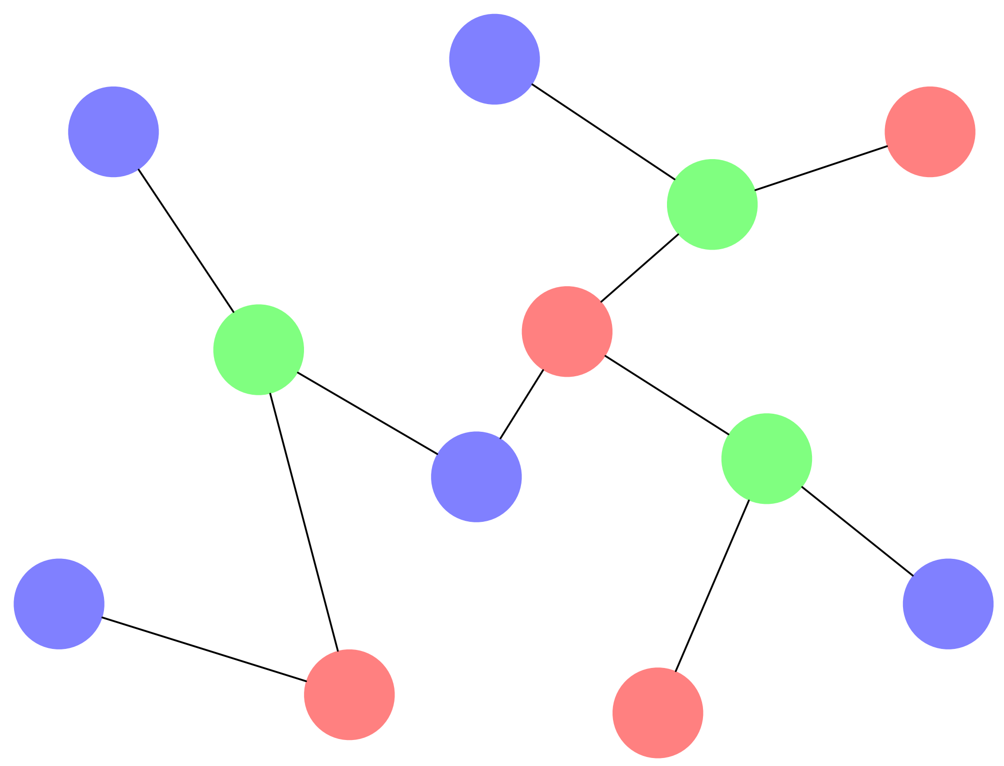

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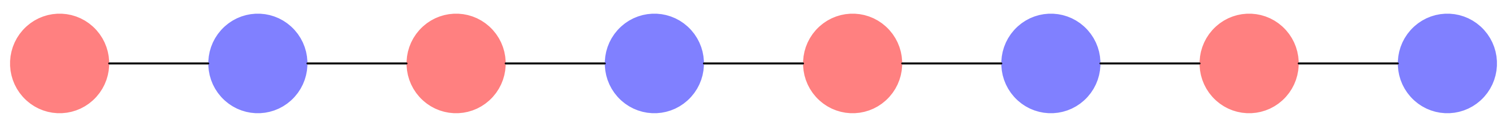

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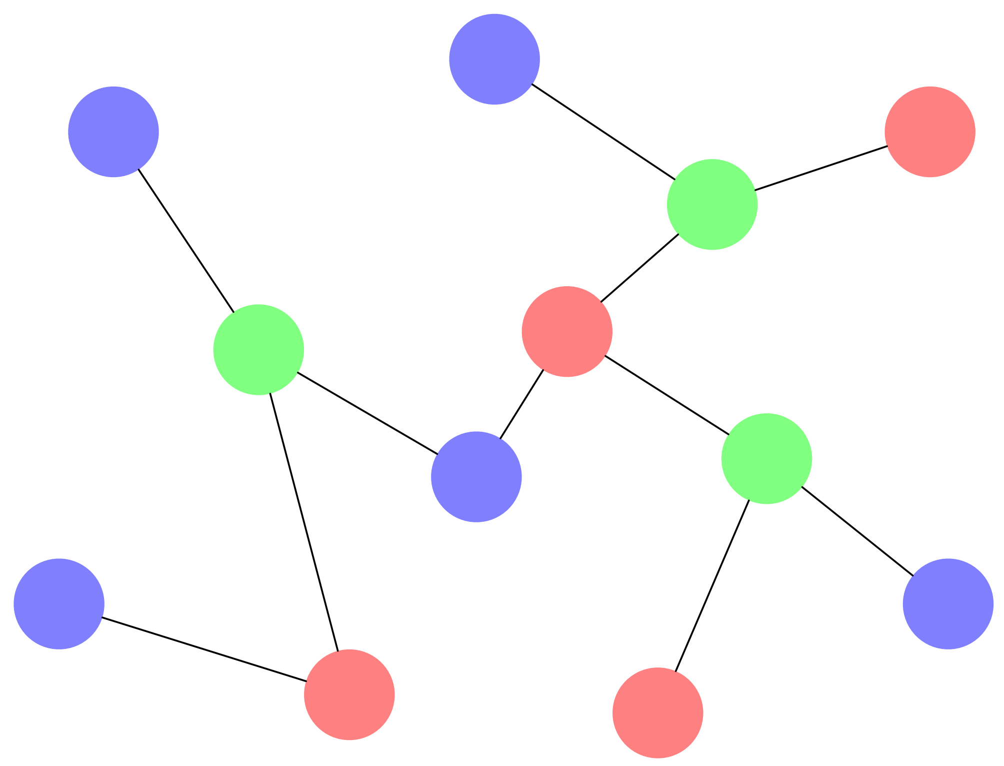

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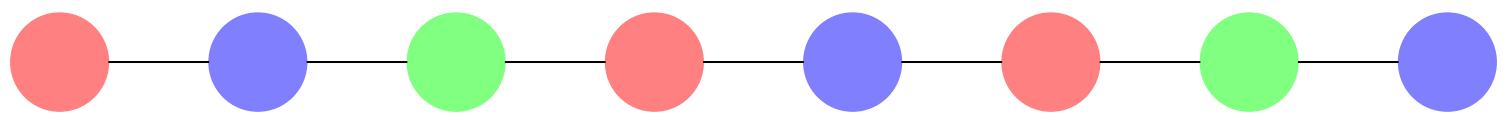

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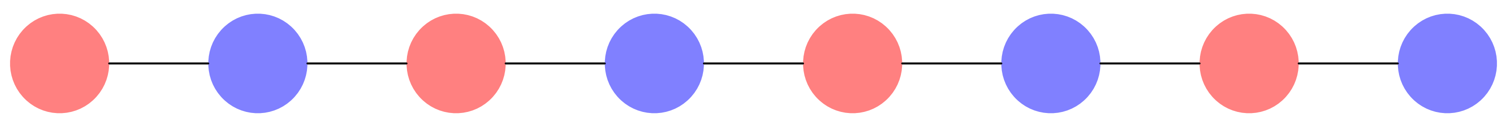

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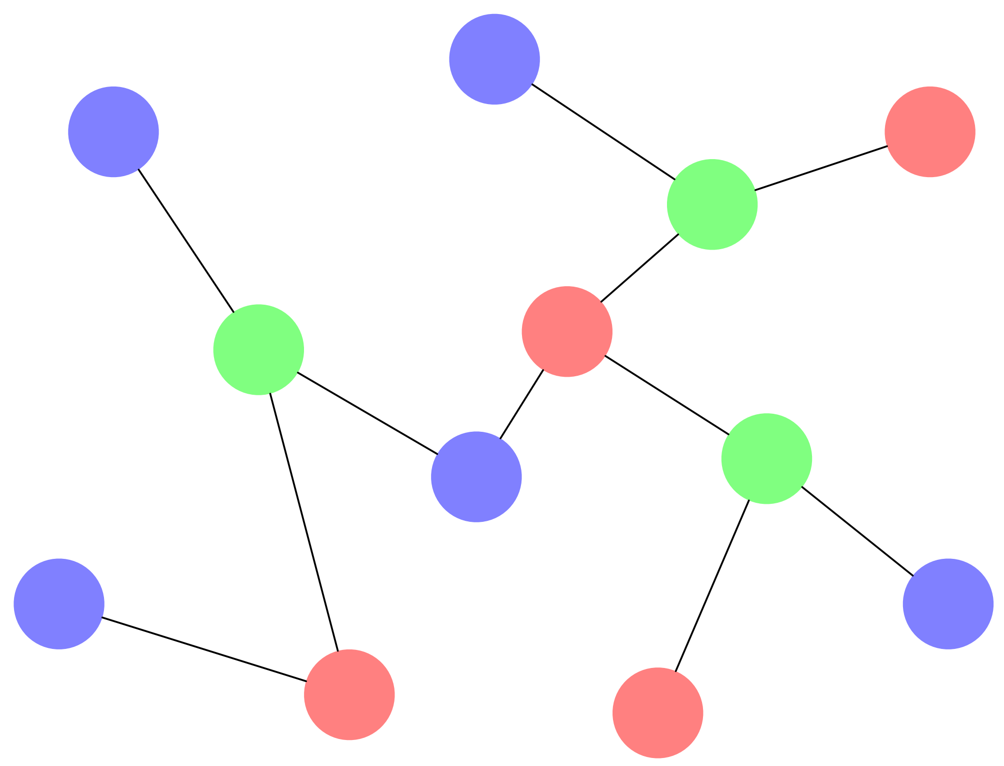

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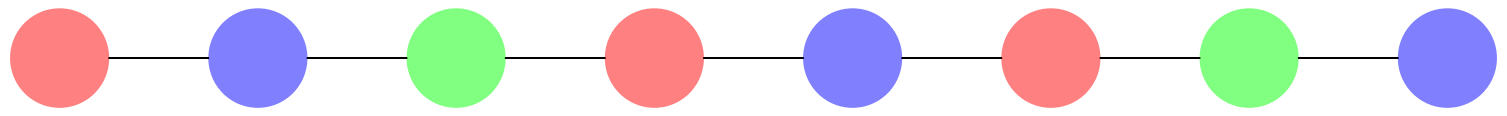

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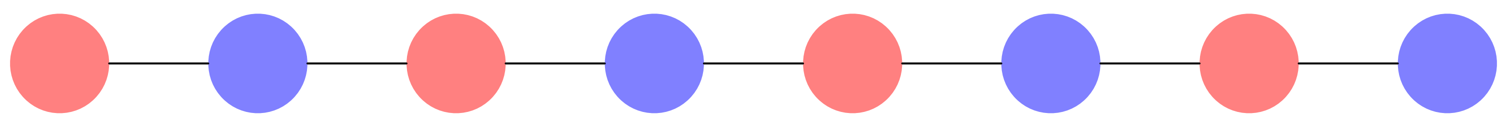

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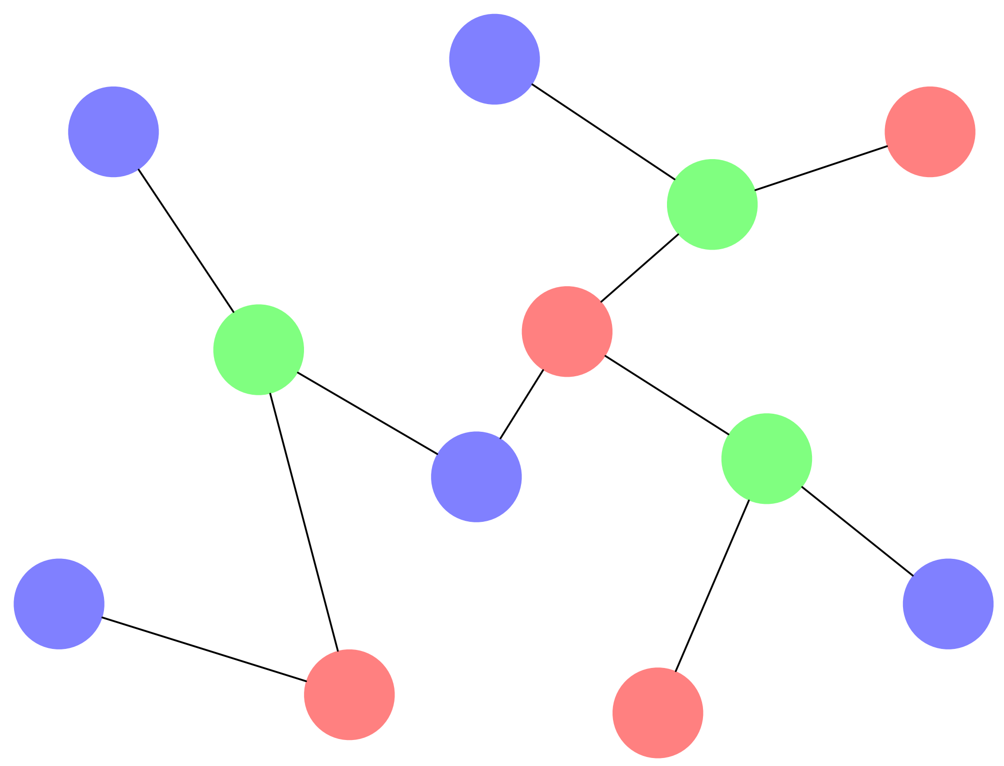

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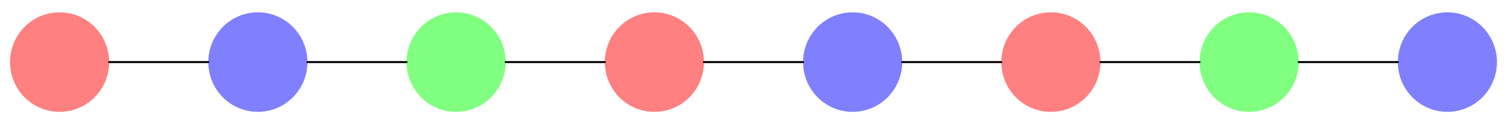

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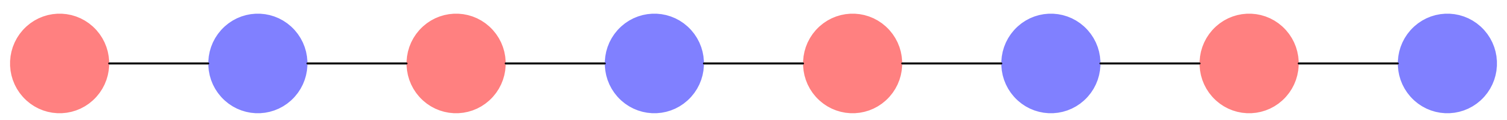

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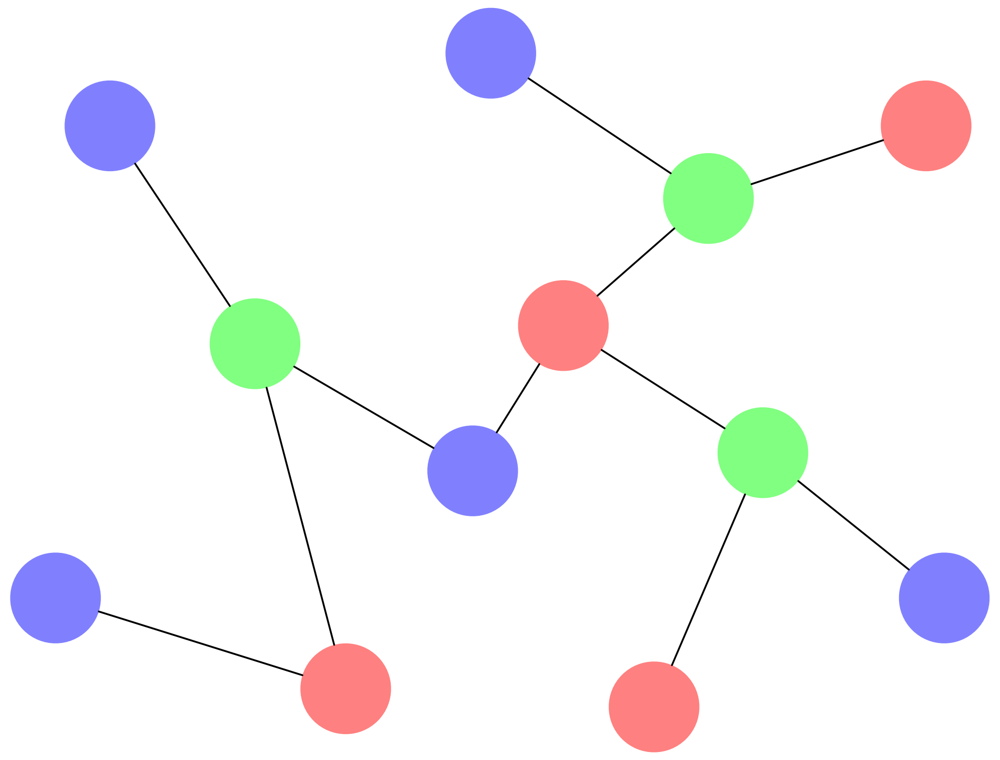

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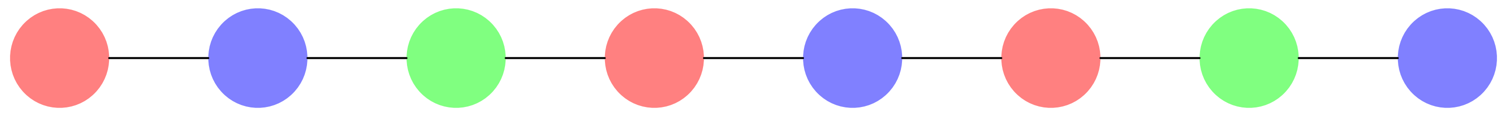

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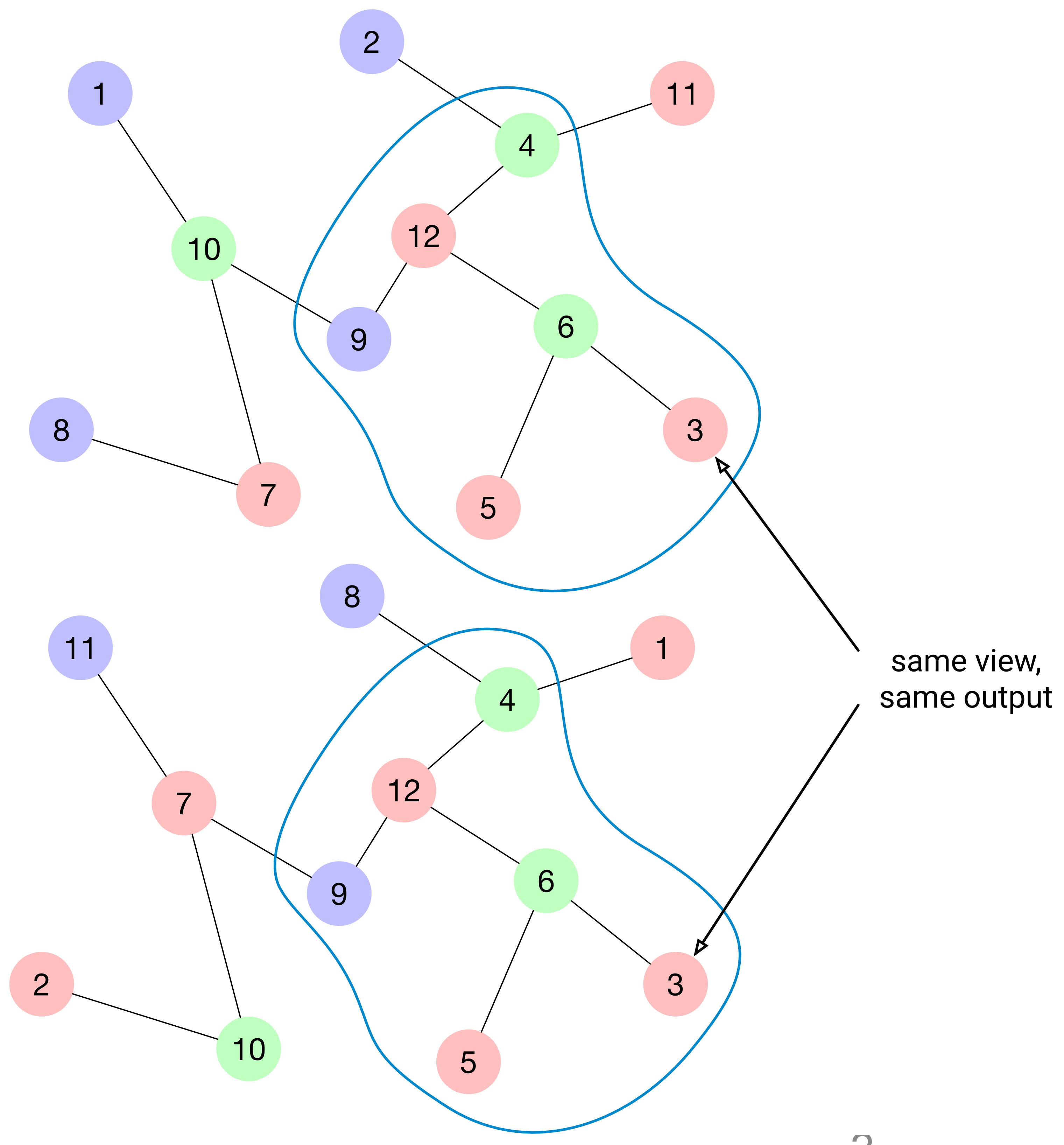

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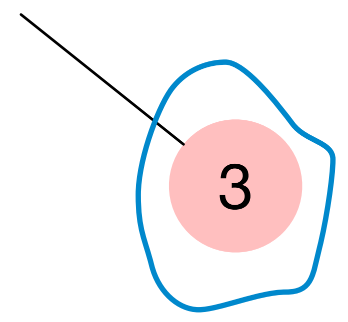

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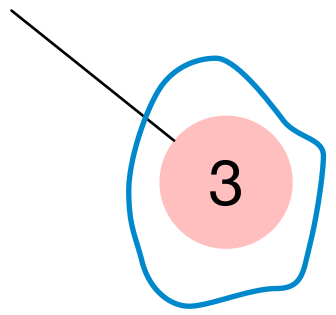

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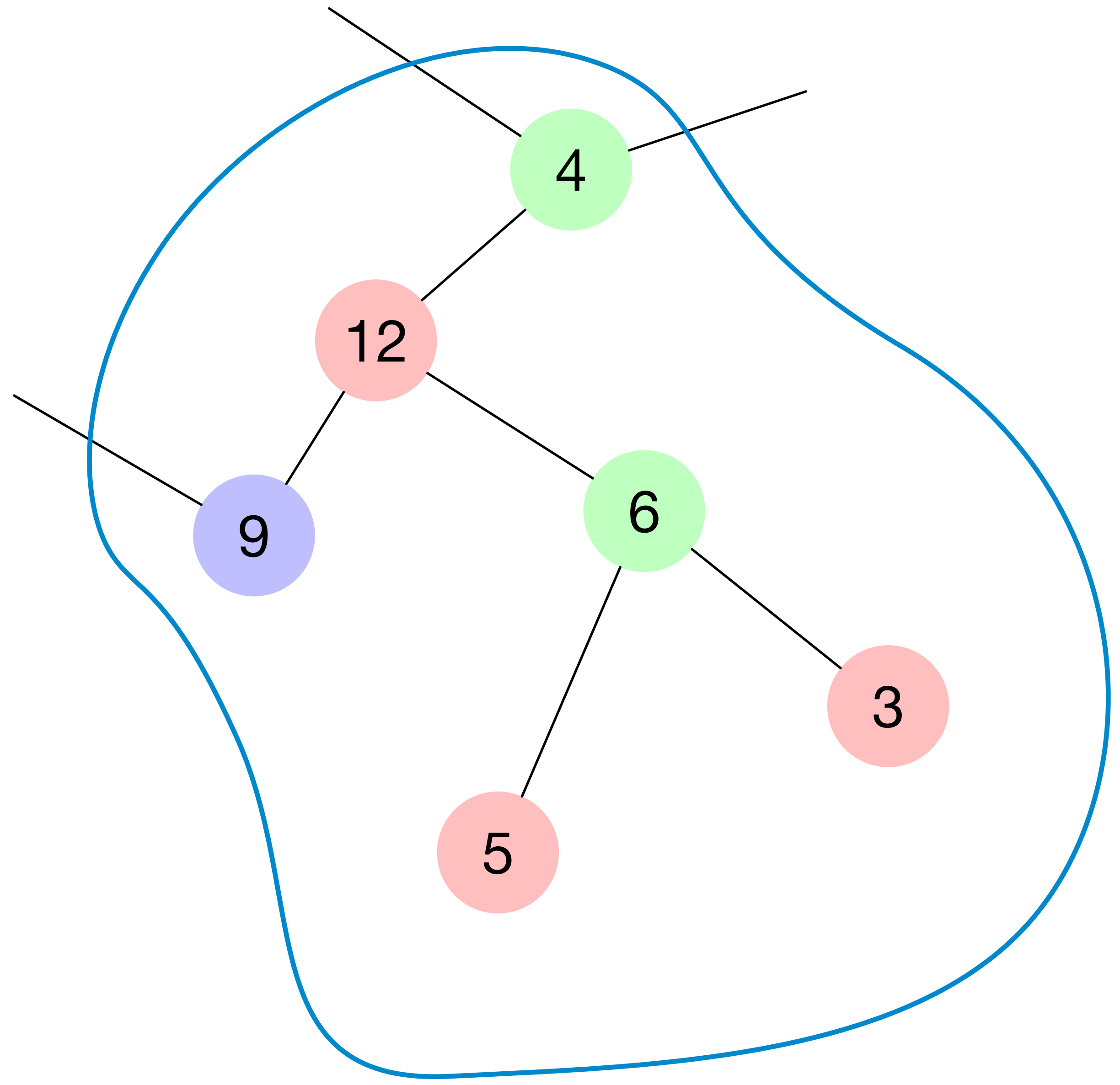

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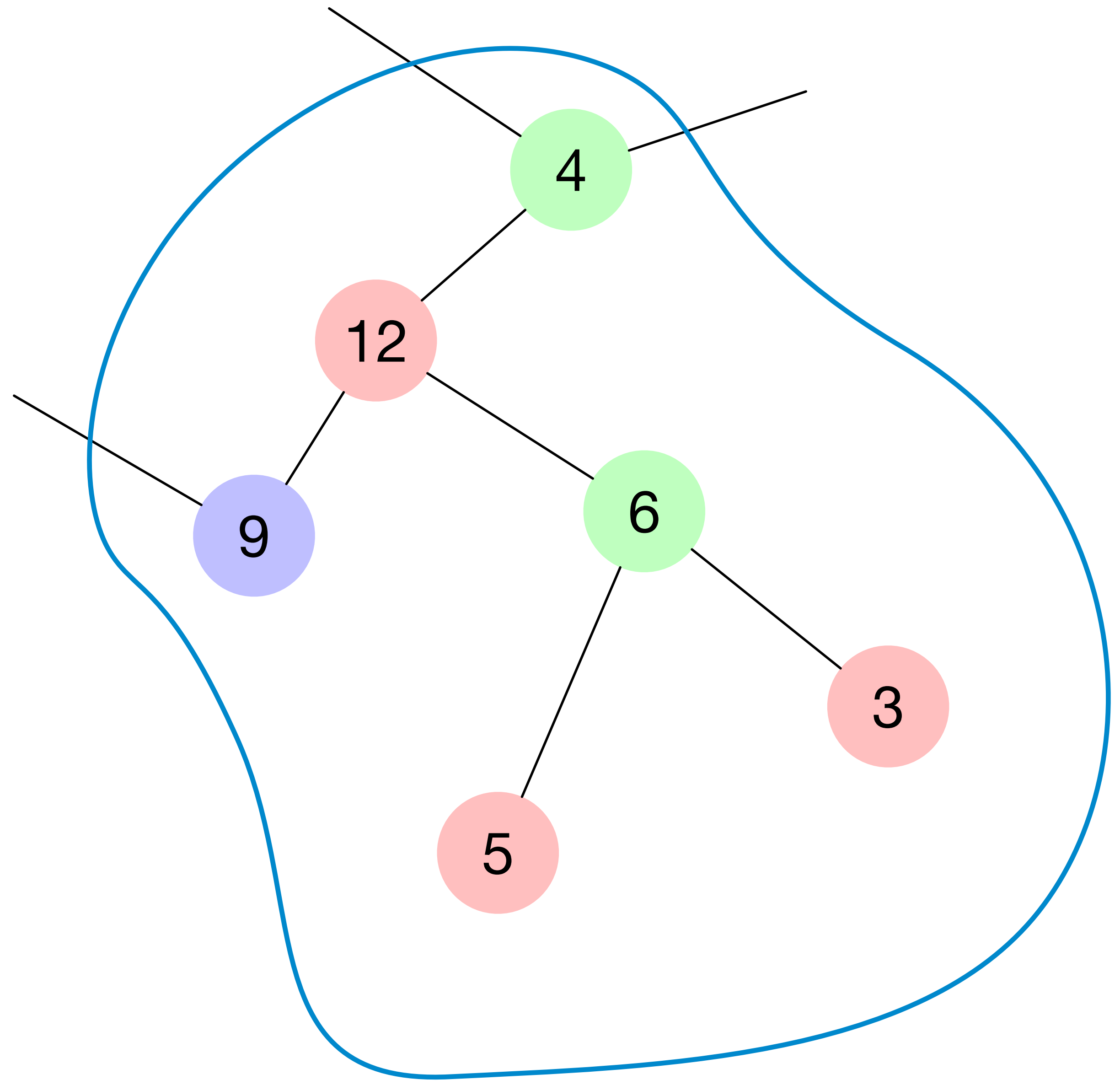

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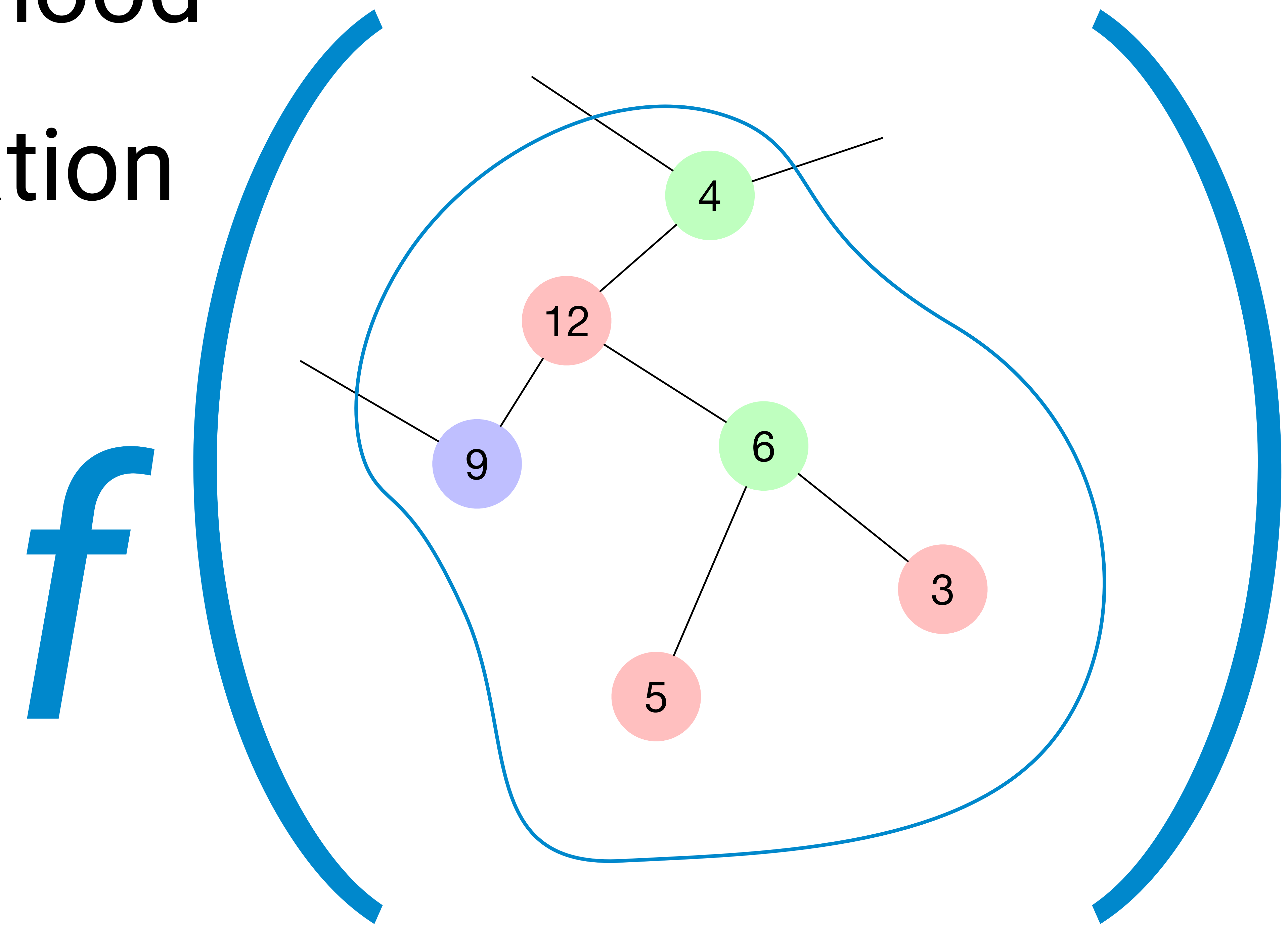

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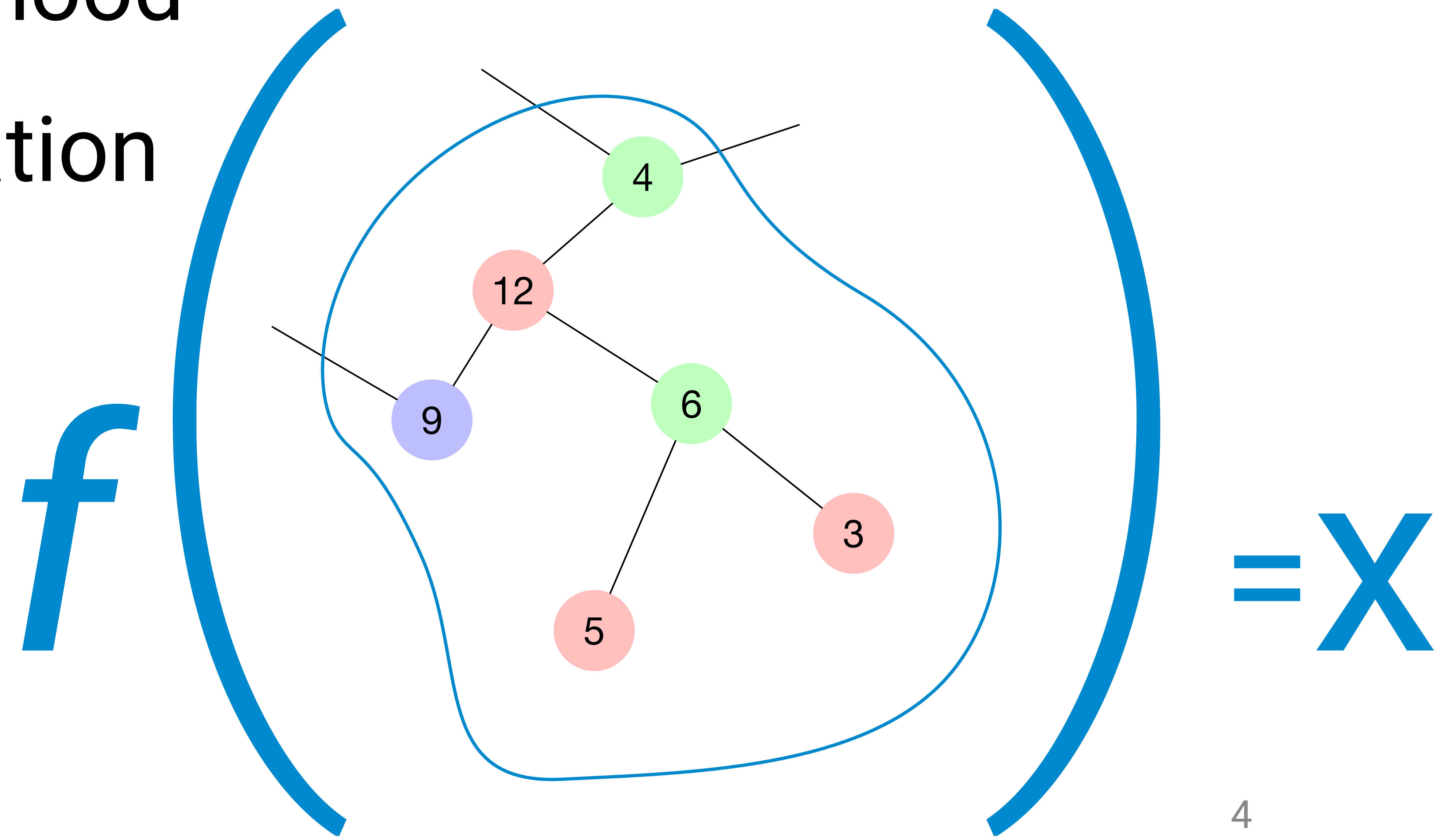

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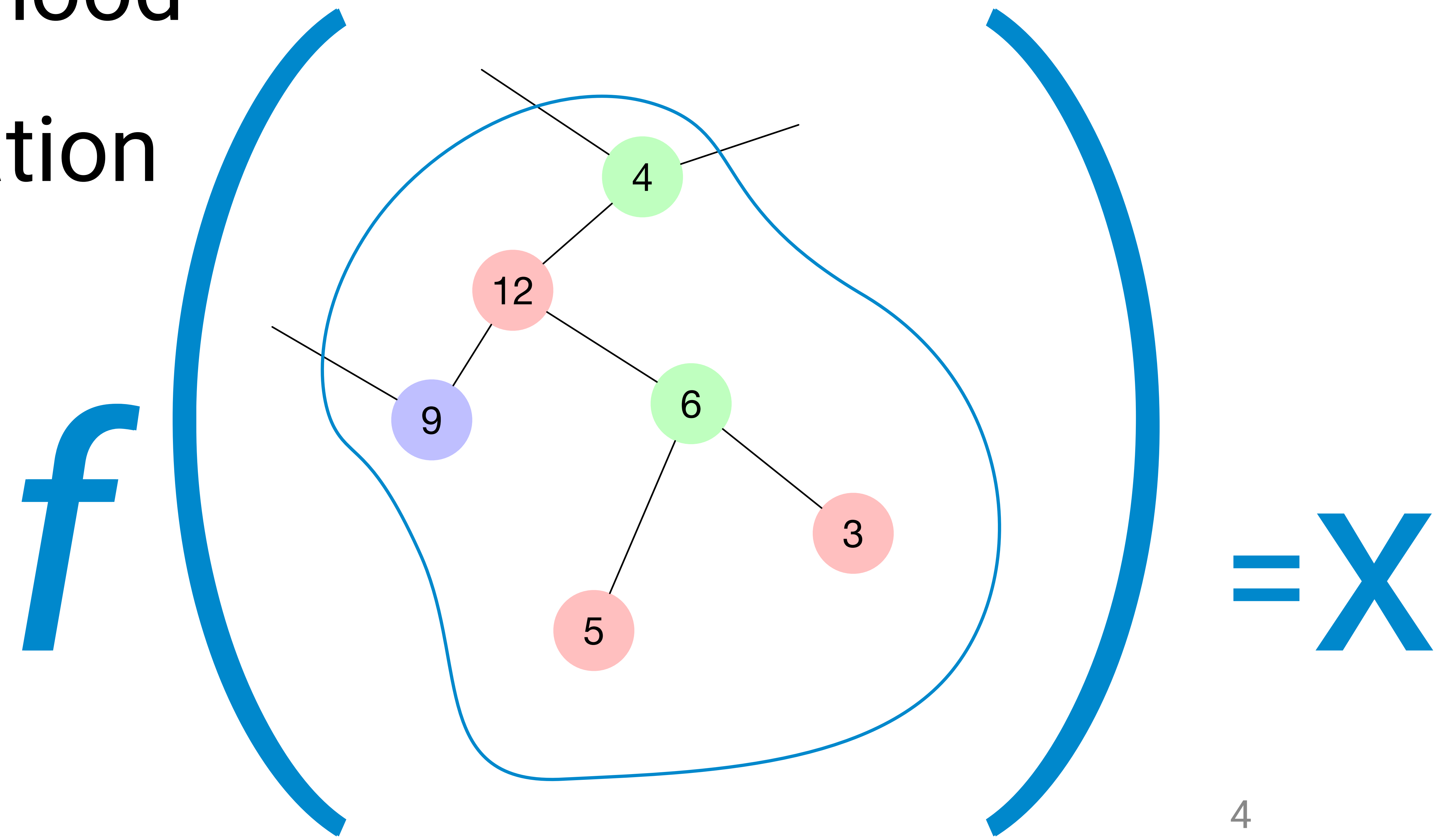
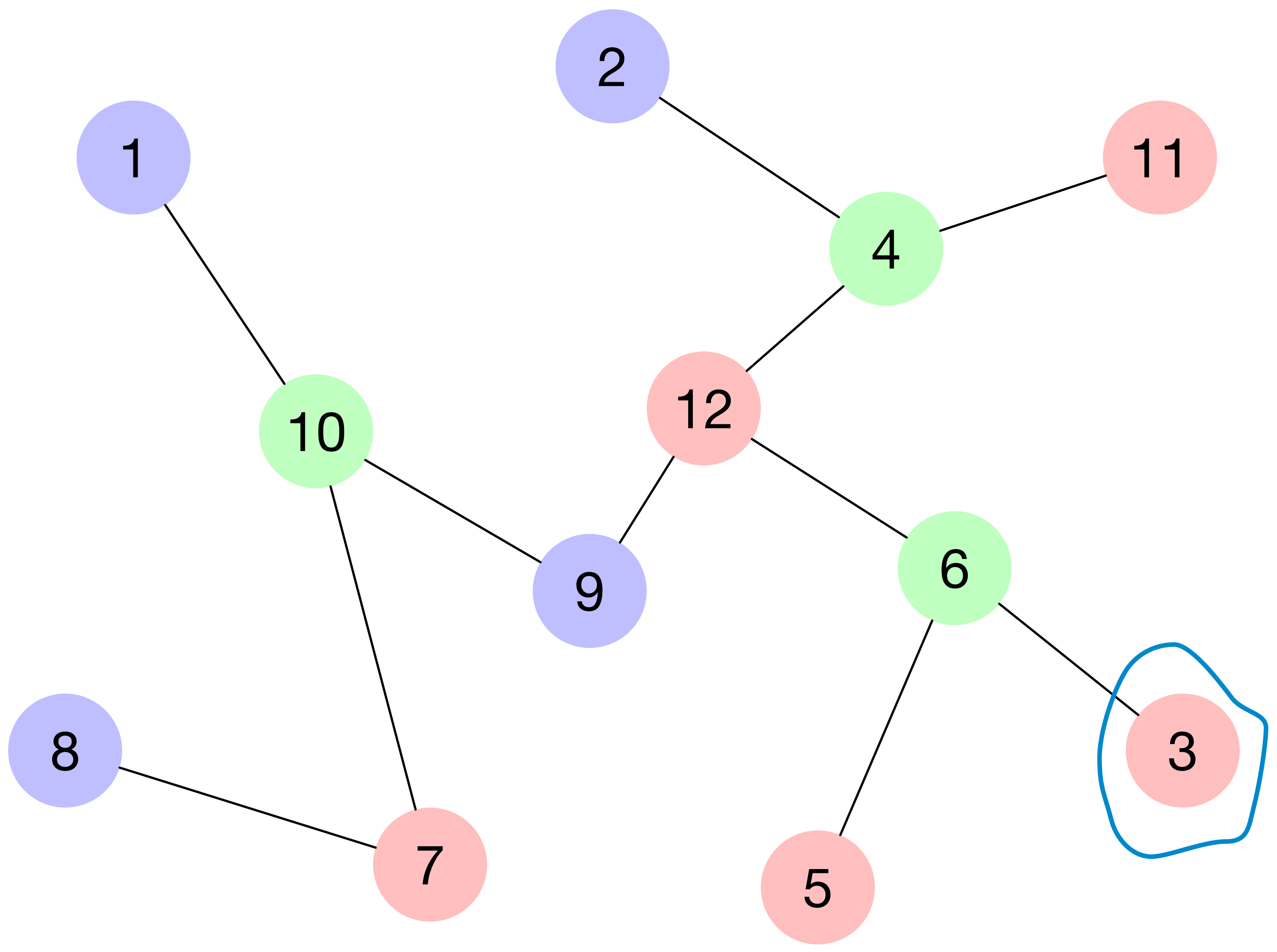


Locality (Example)

A 0-round algorithm is just a mapping from input to output



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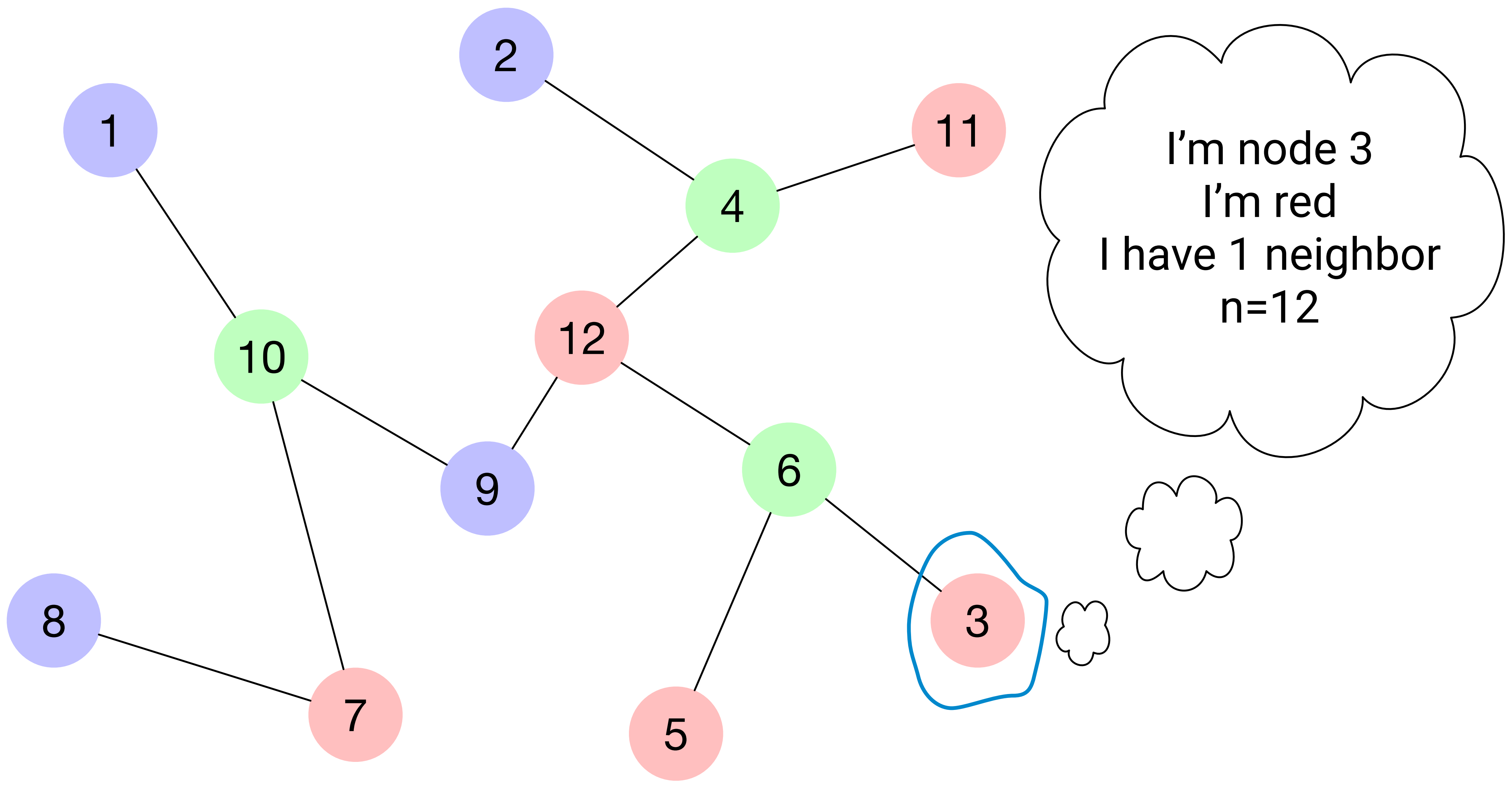

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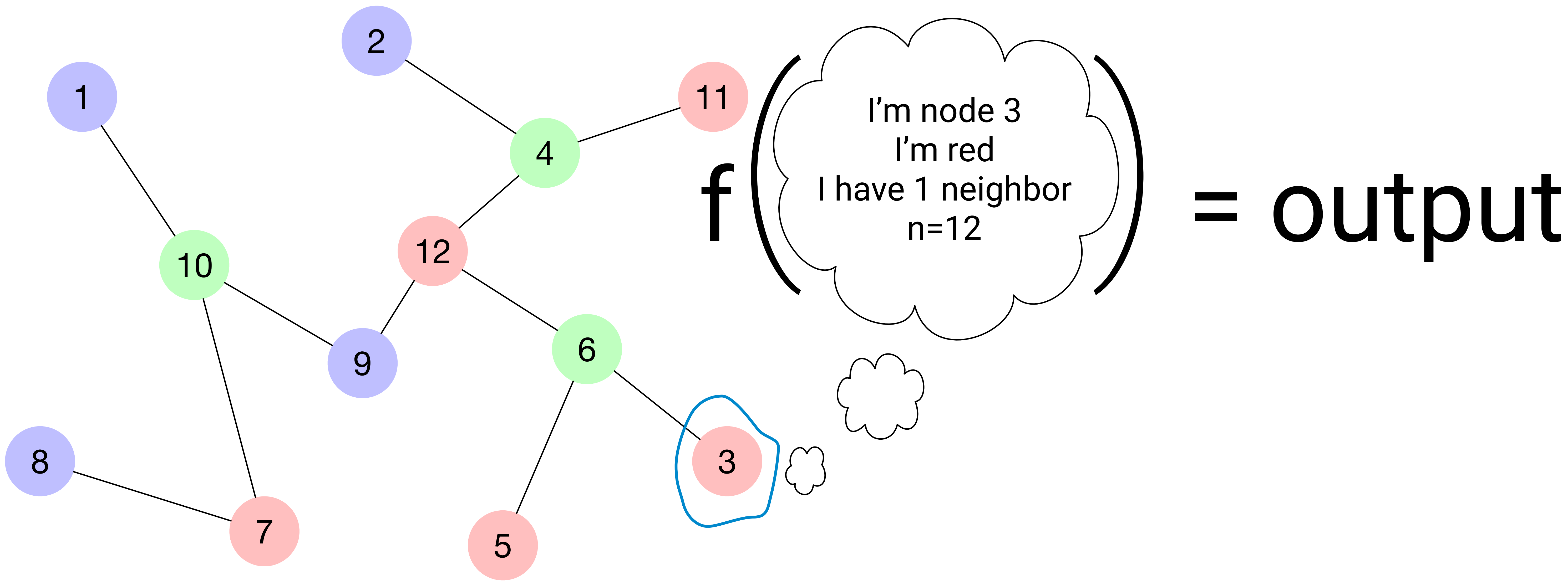

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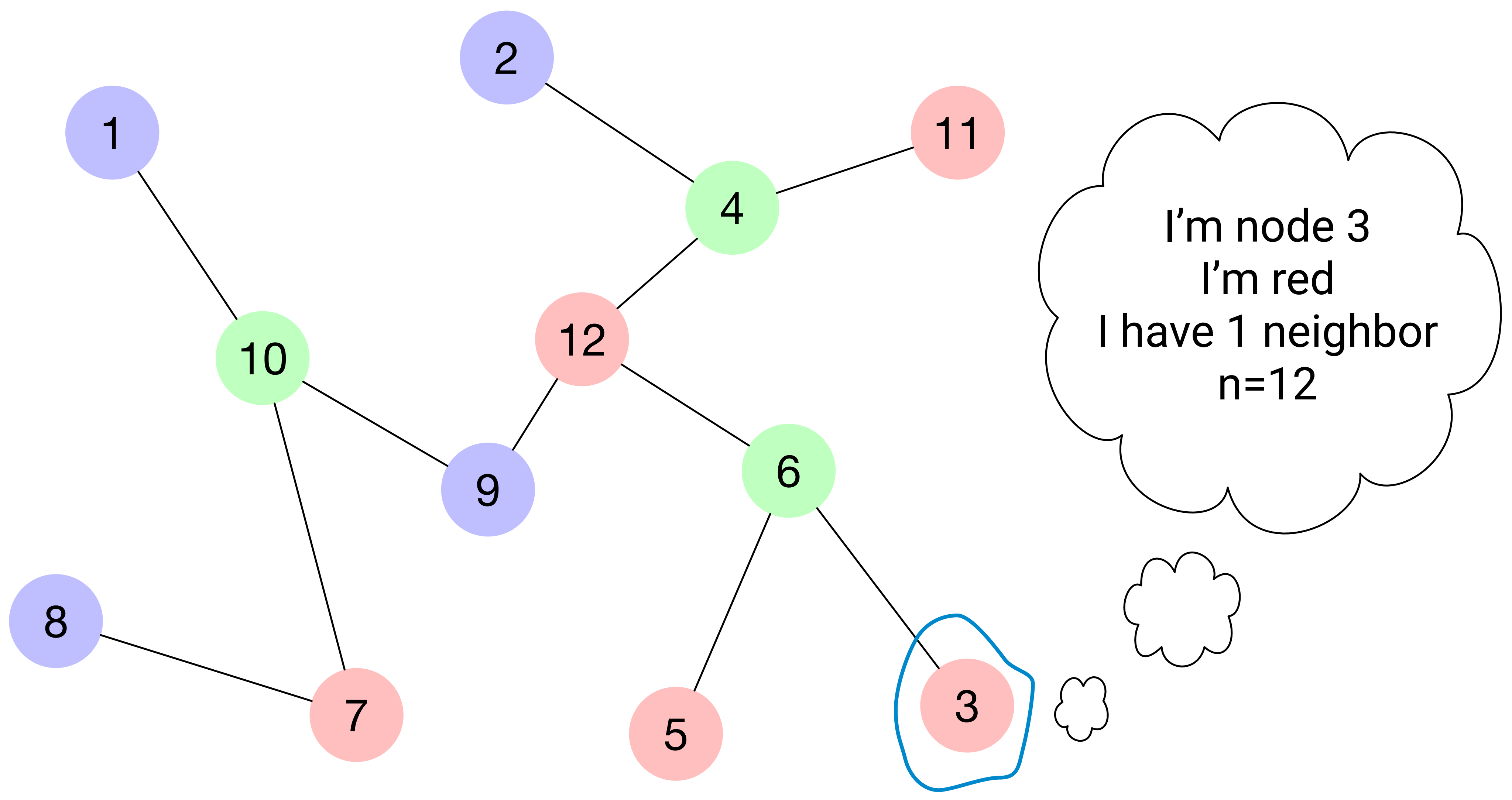

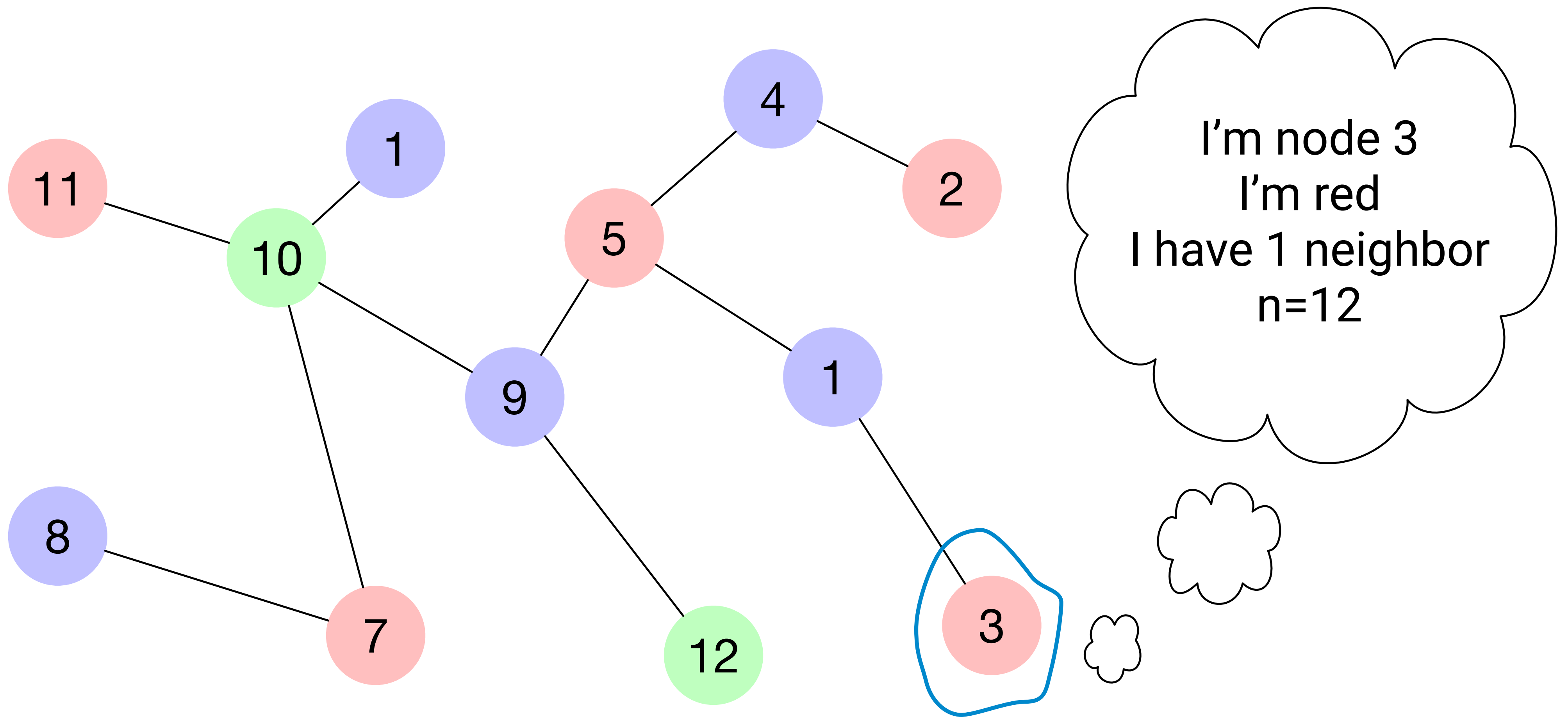

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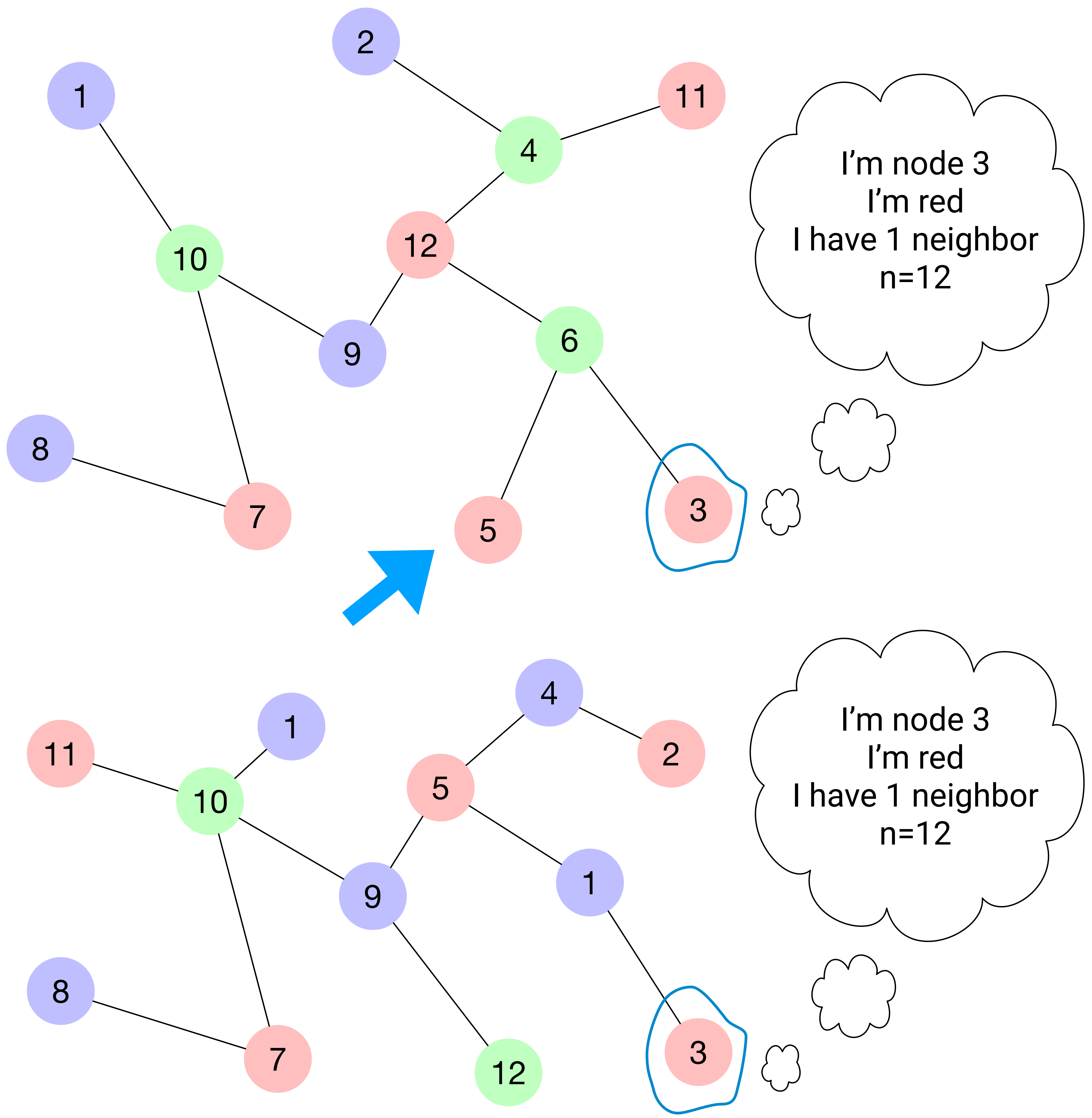

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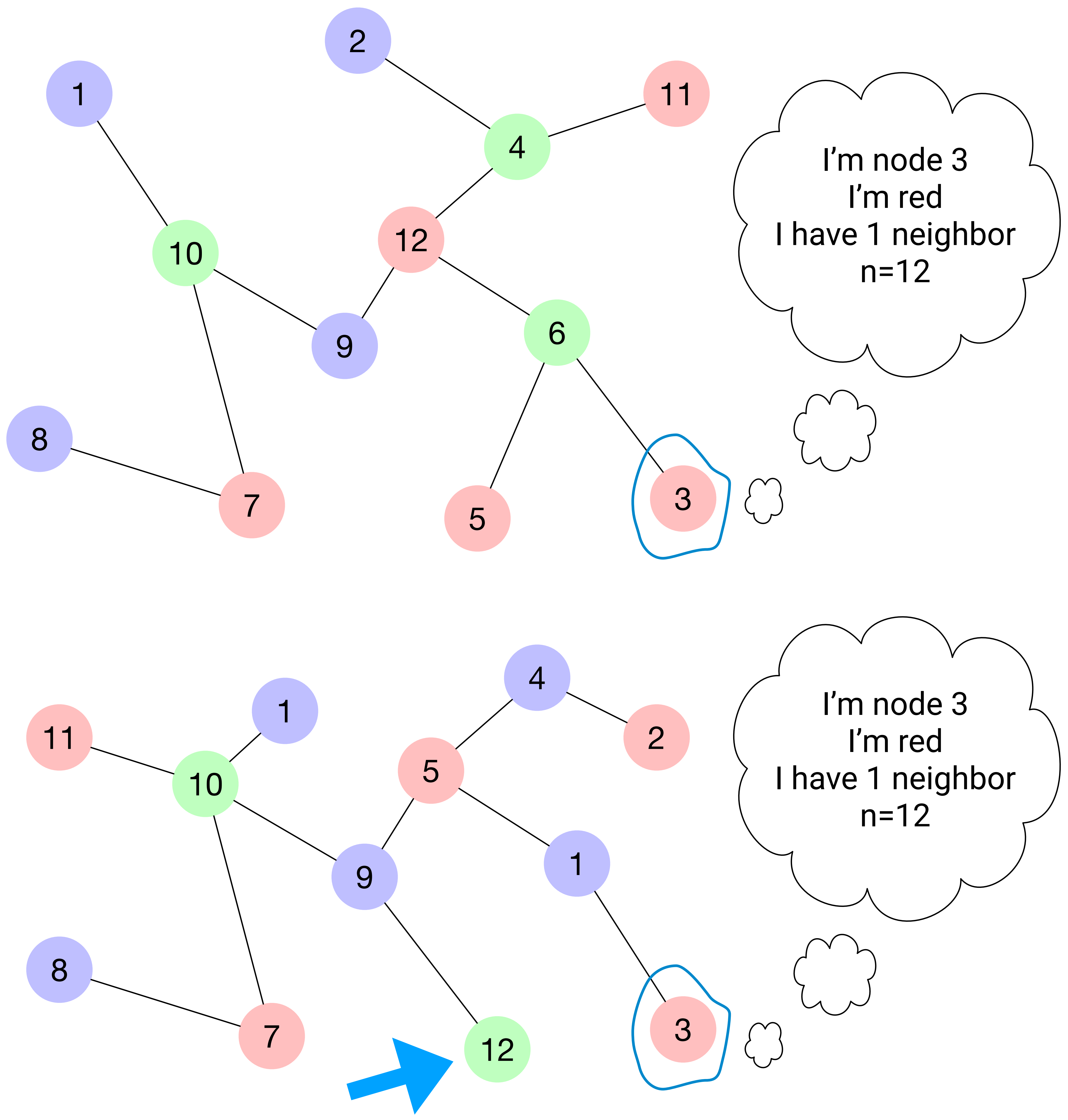

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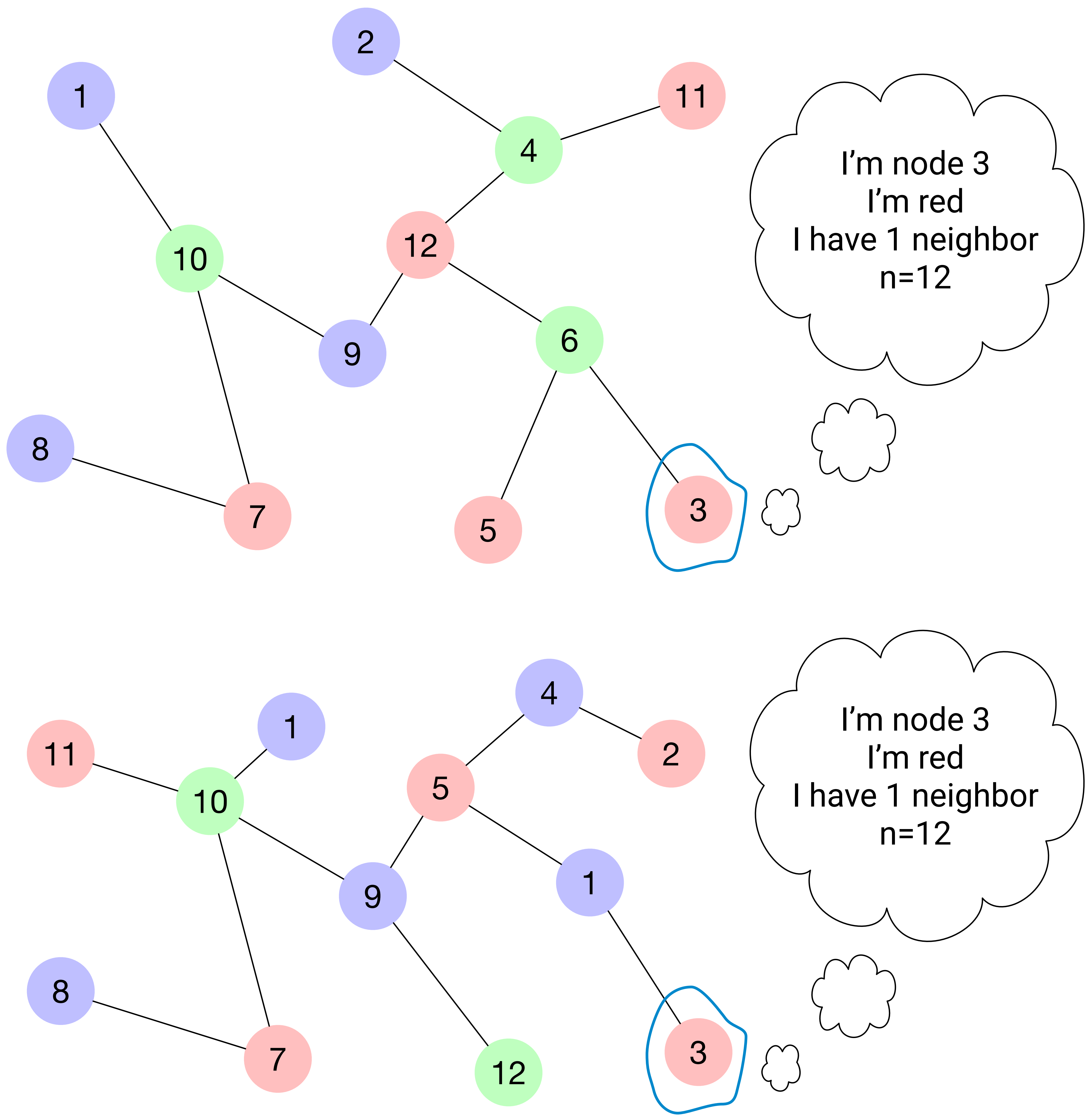

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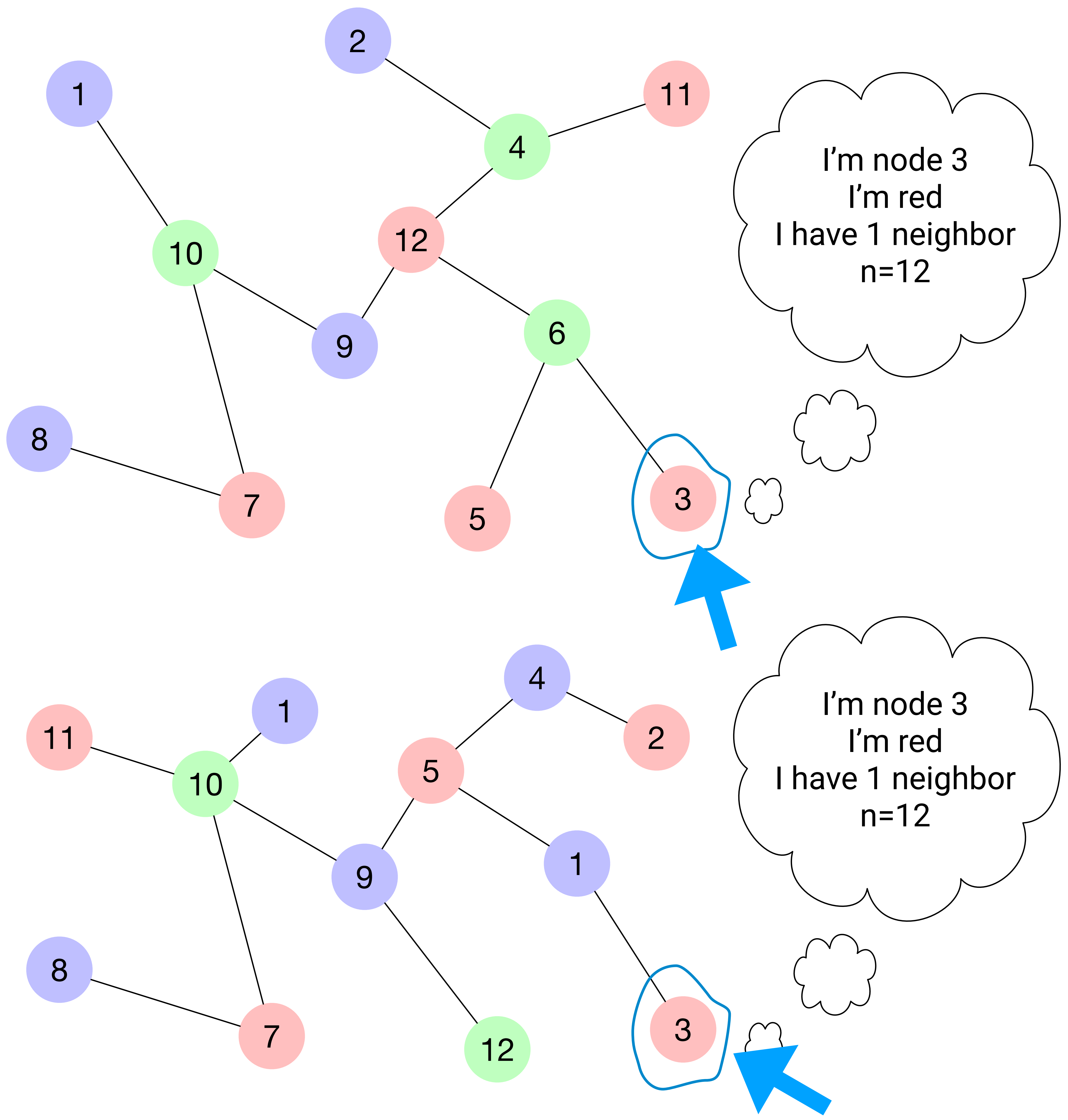

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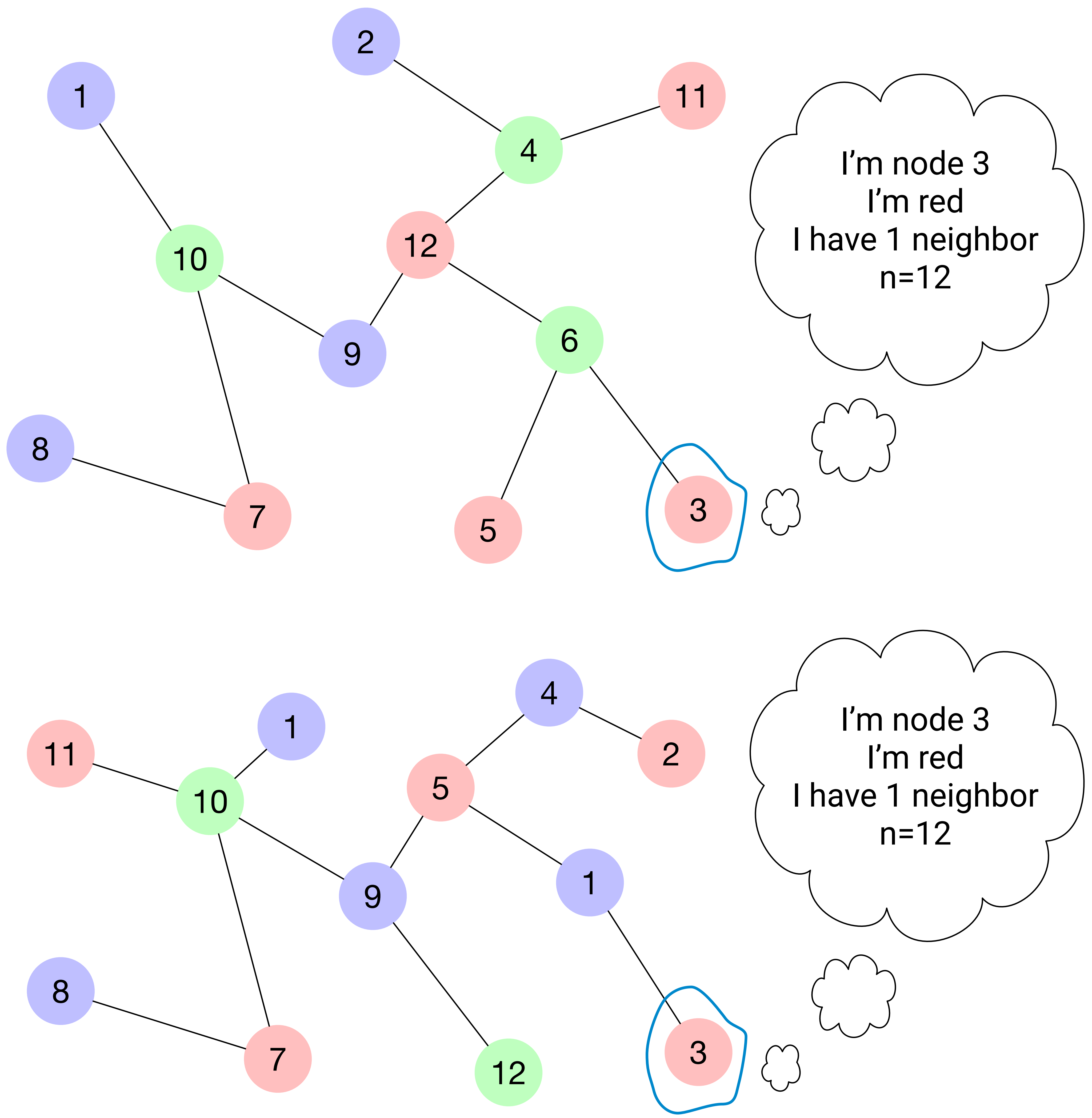

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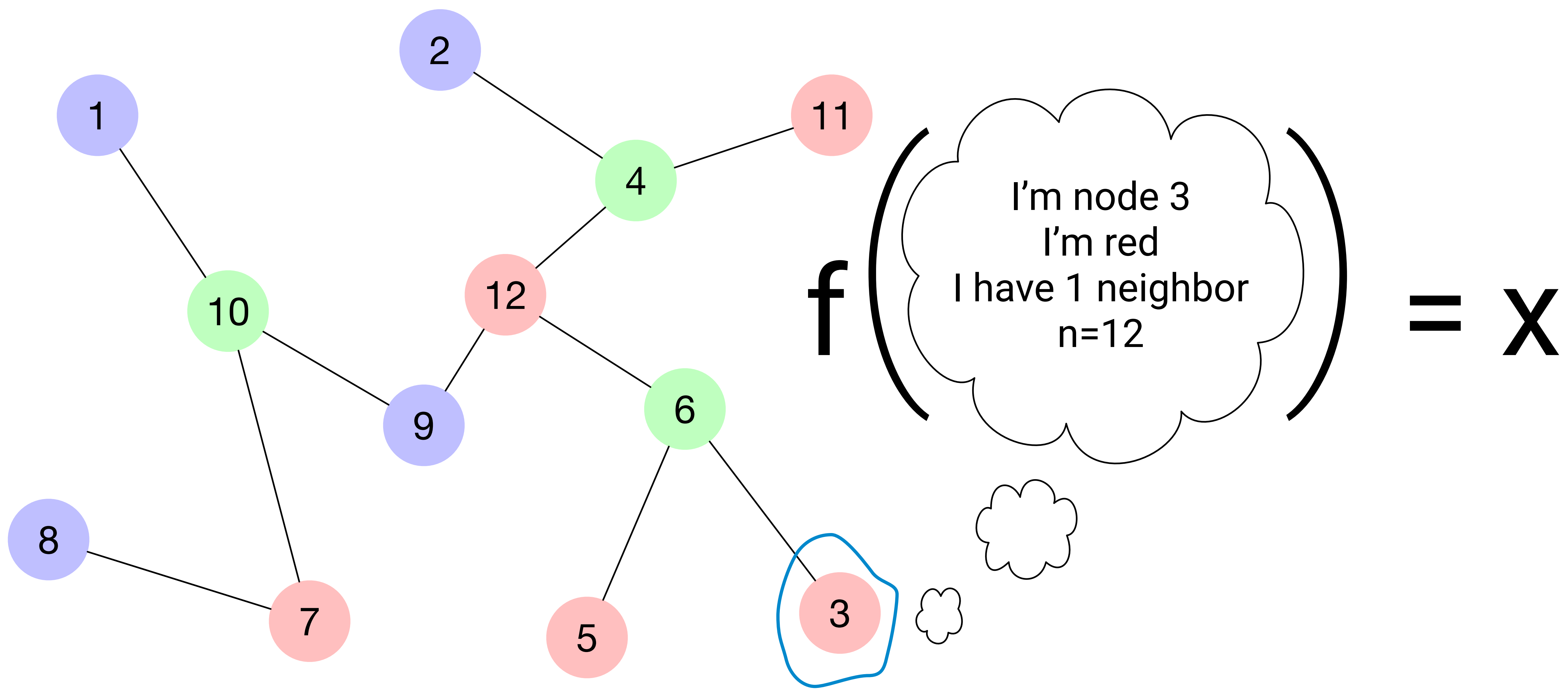

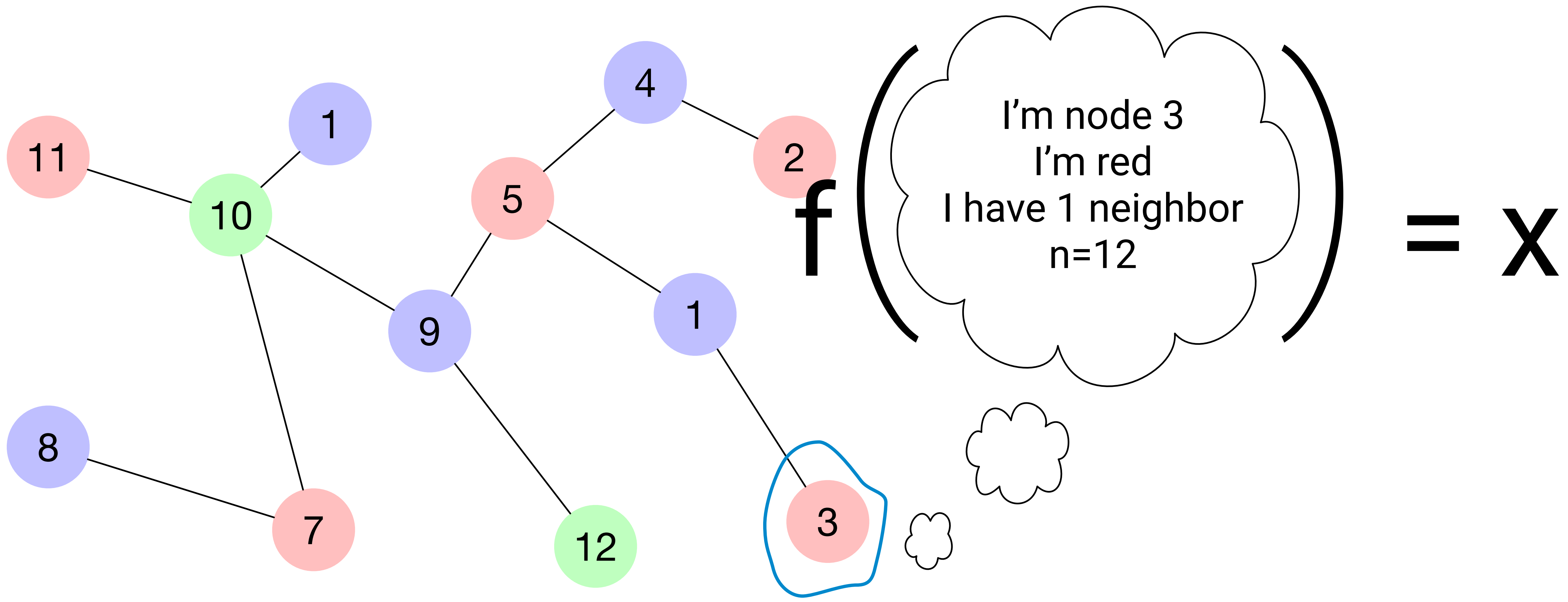

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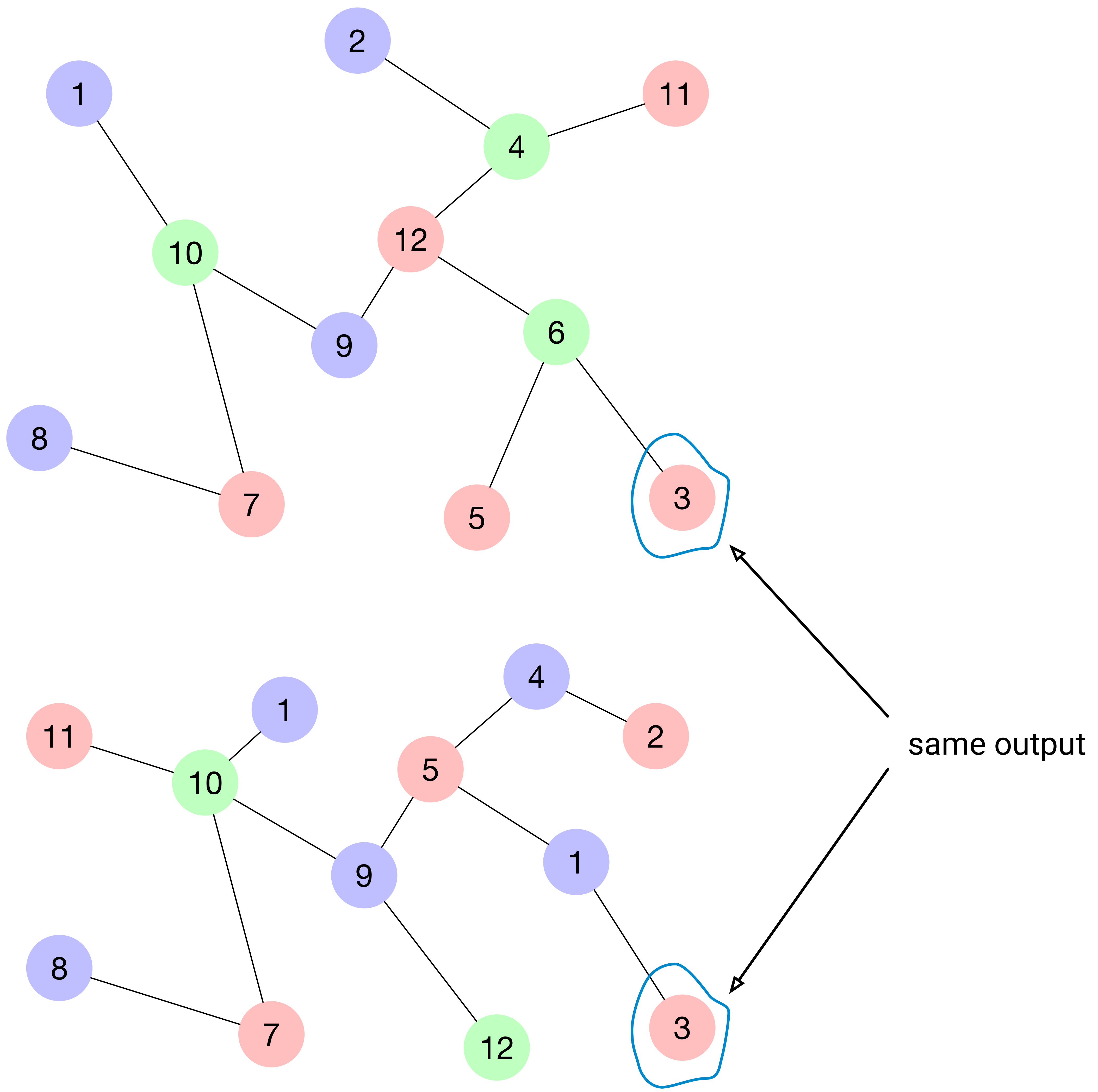

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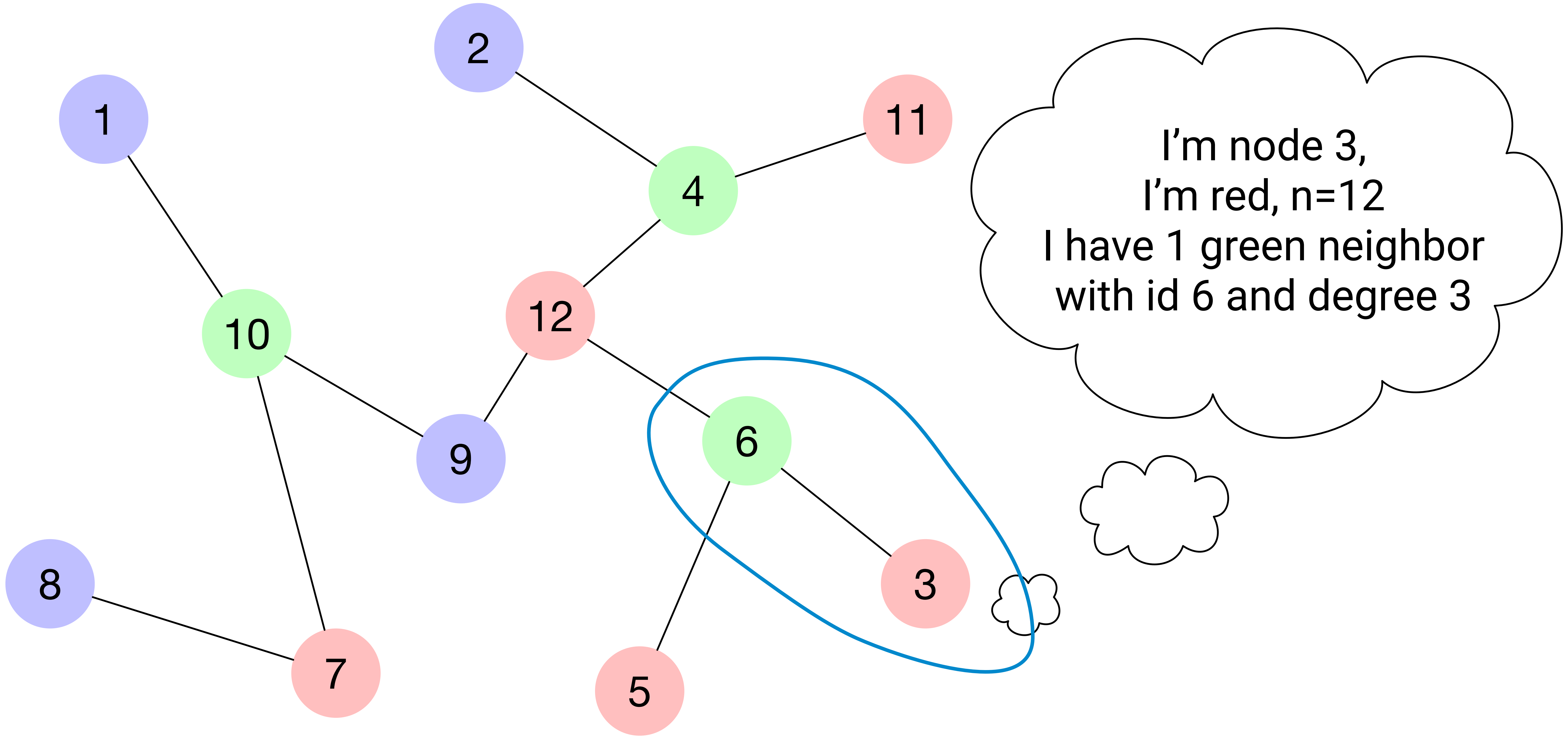

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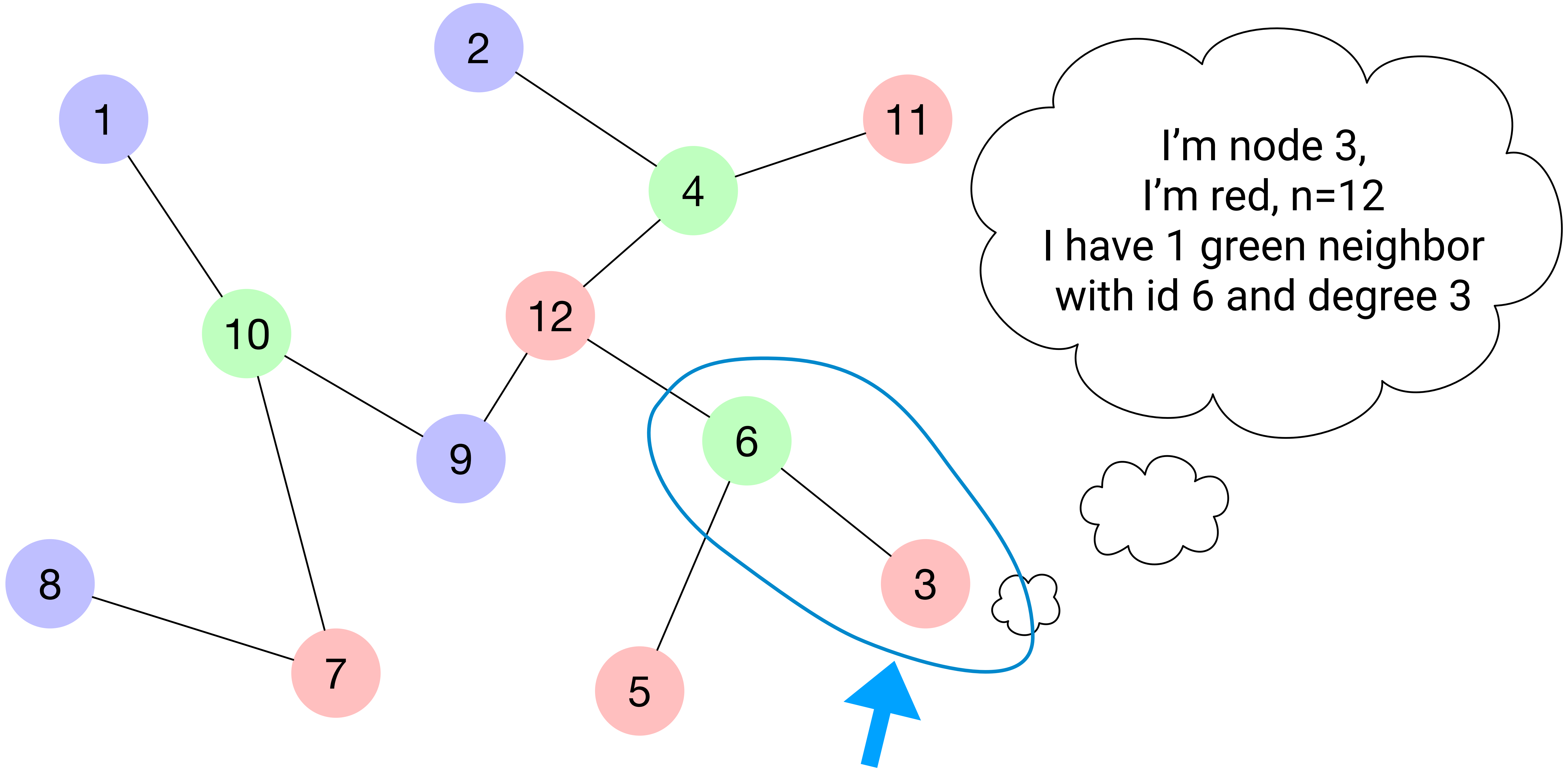

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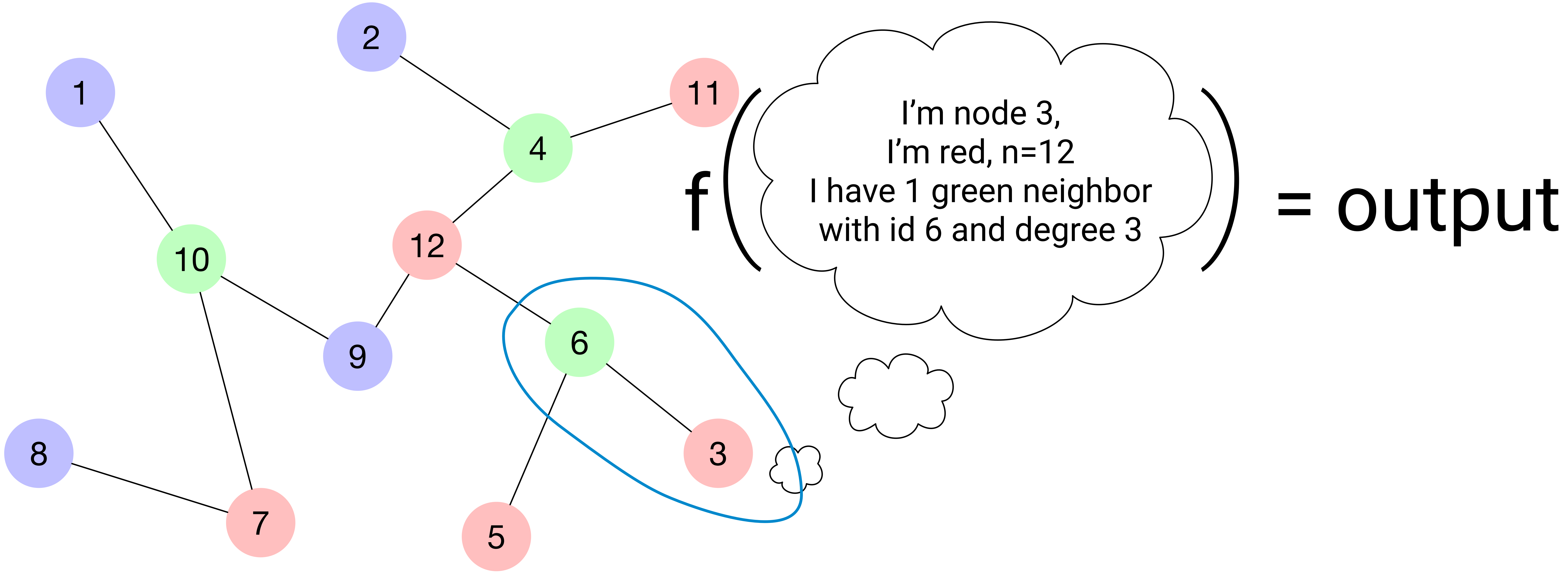

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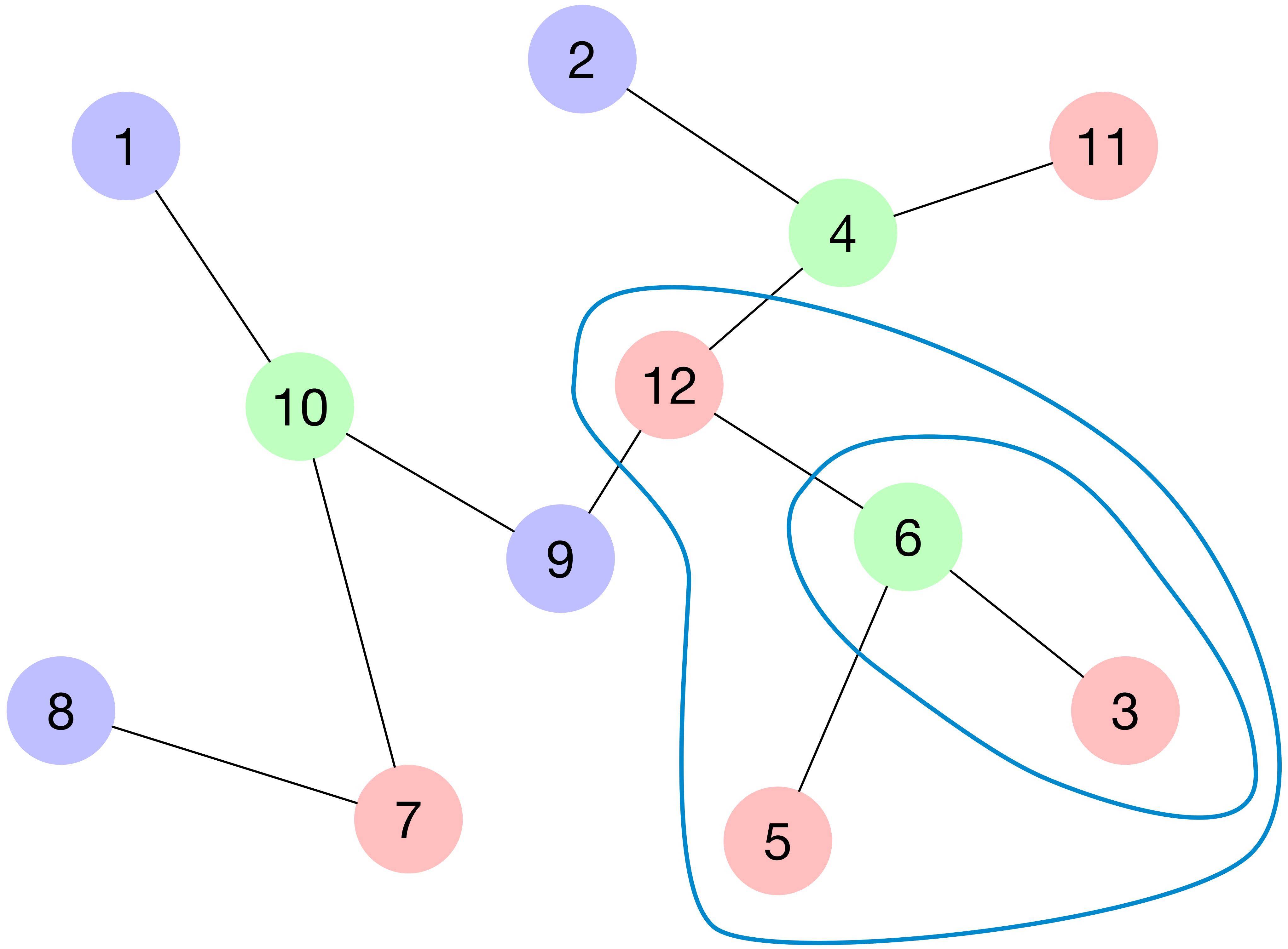

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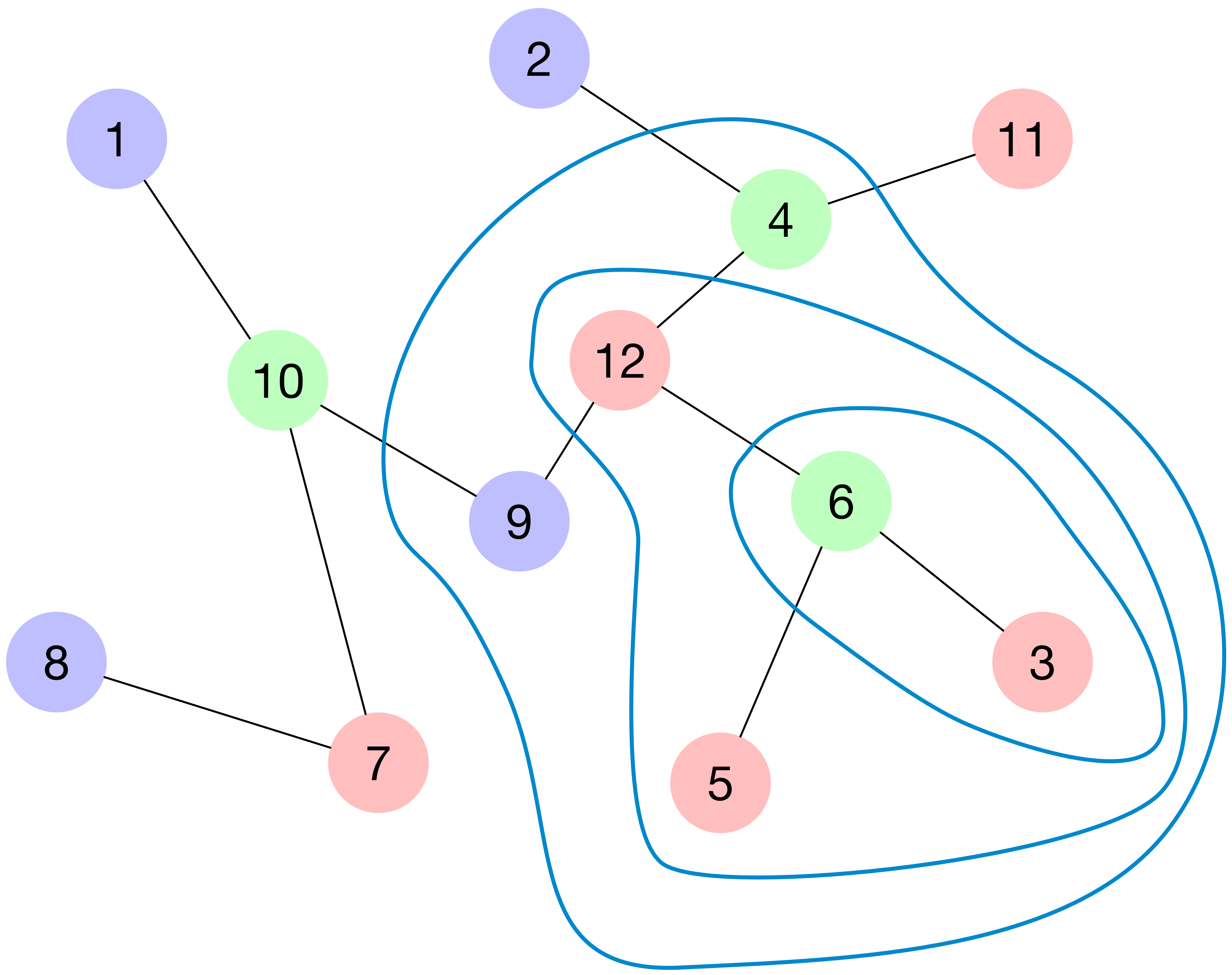

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Locality

A T-round algorithm is just a mapping from radius-T balls to outputs.

Proof by induction.

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contained in the T-radius neighborhood of v

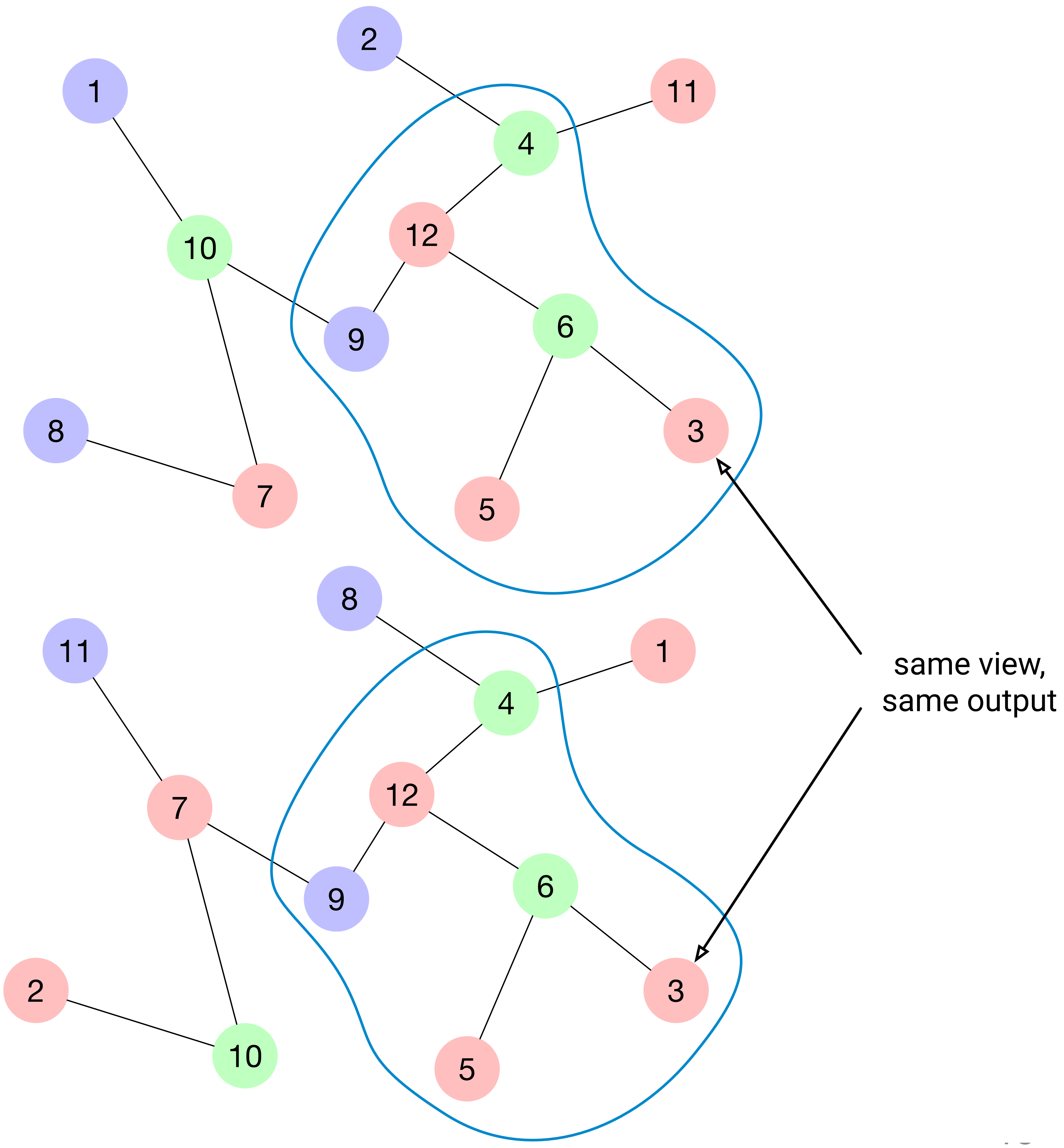
Main technique to prove lower bounds

same radius-T neighborhood

⇩

any T-round algorithm

outputs the same



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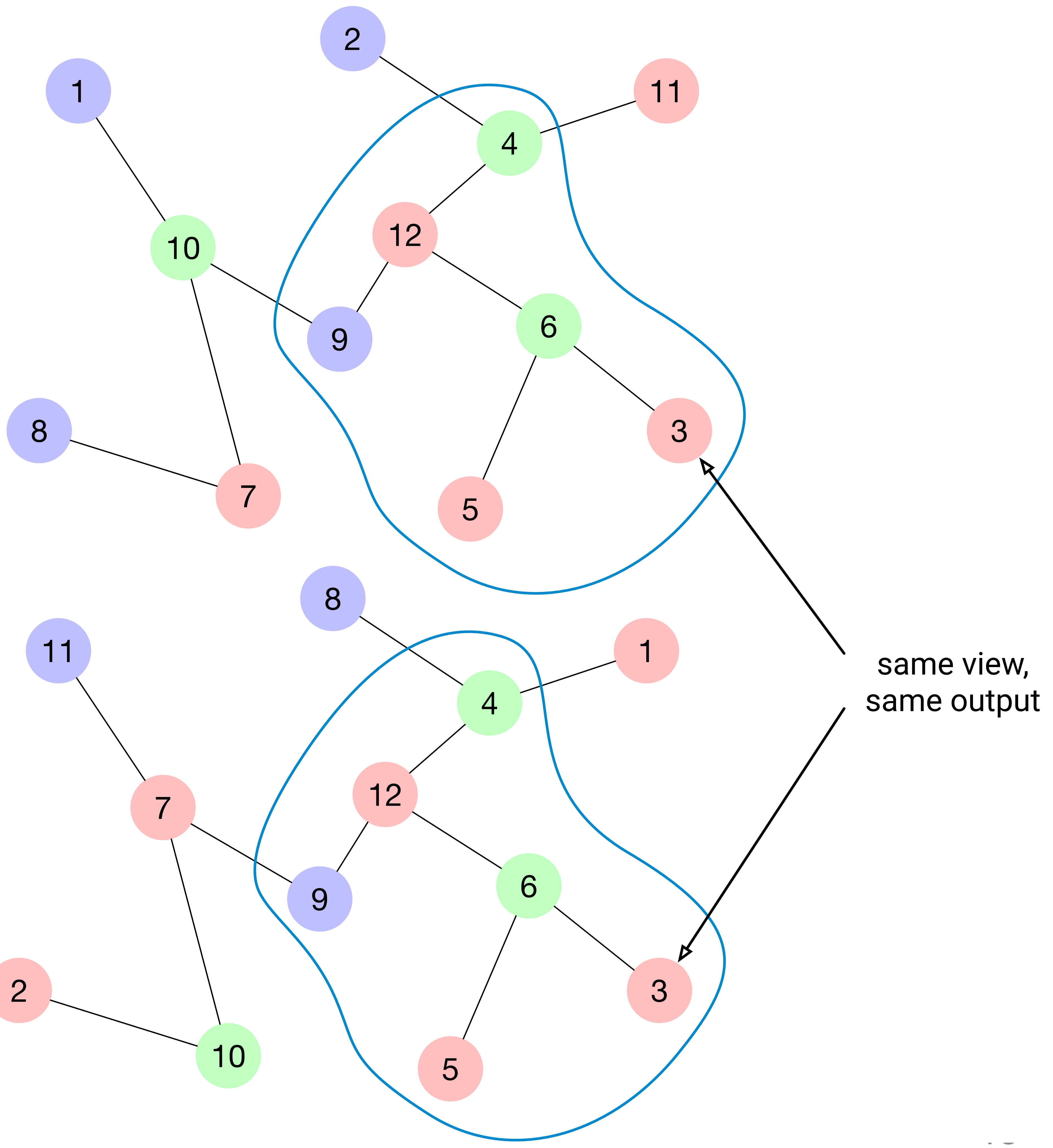

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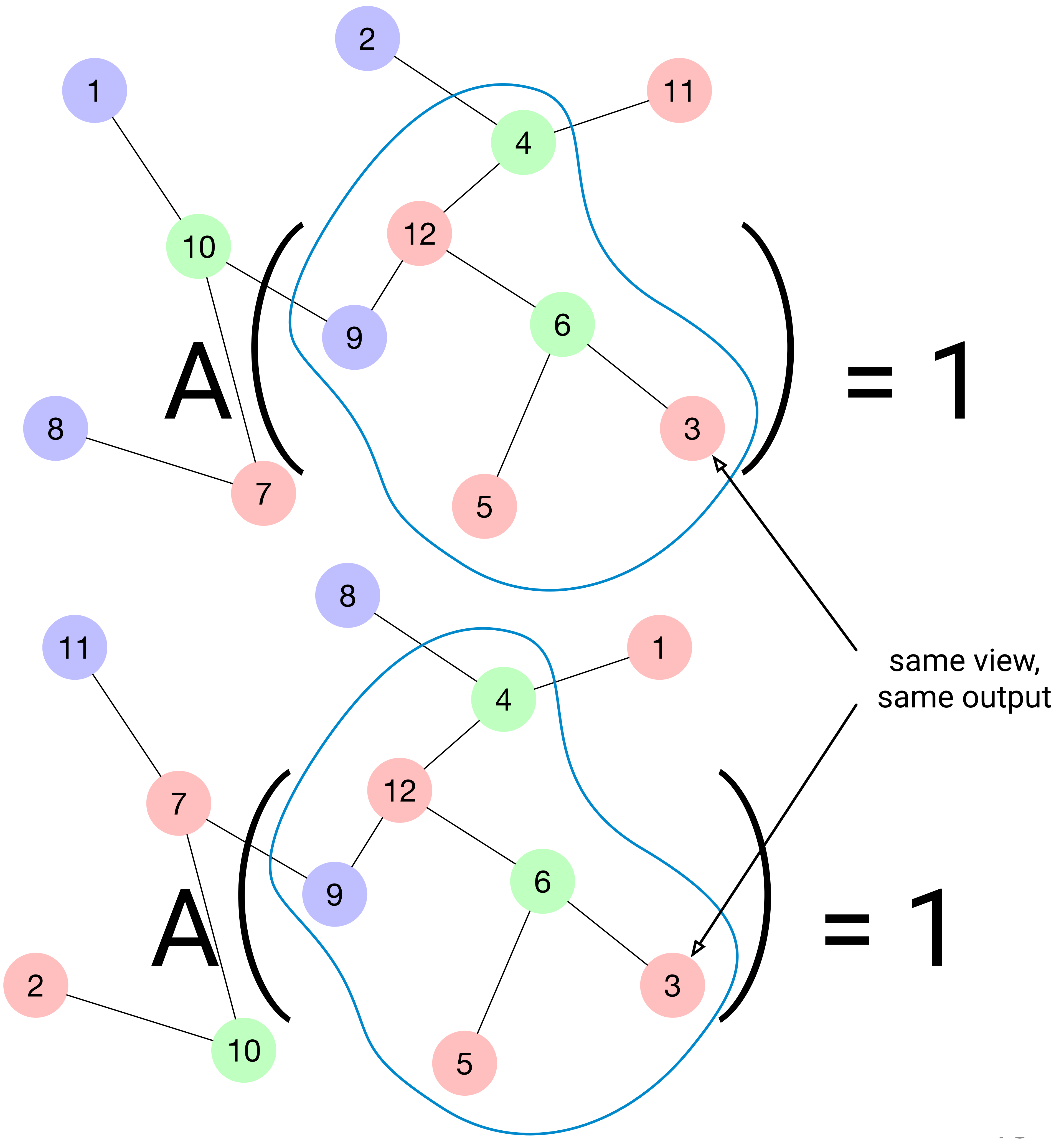

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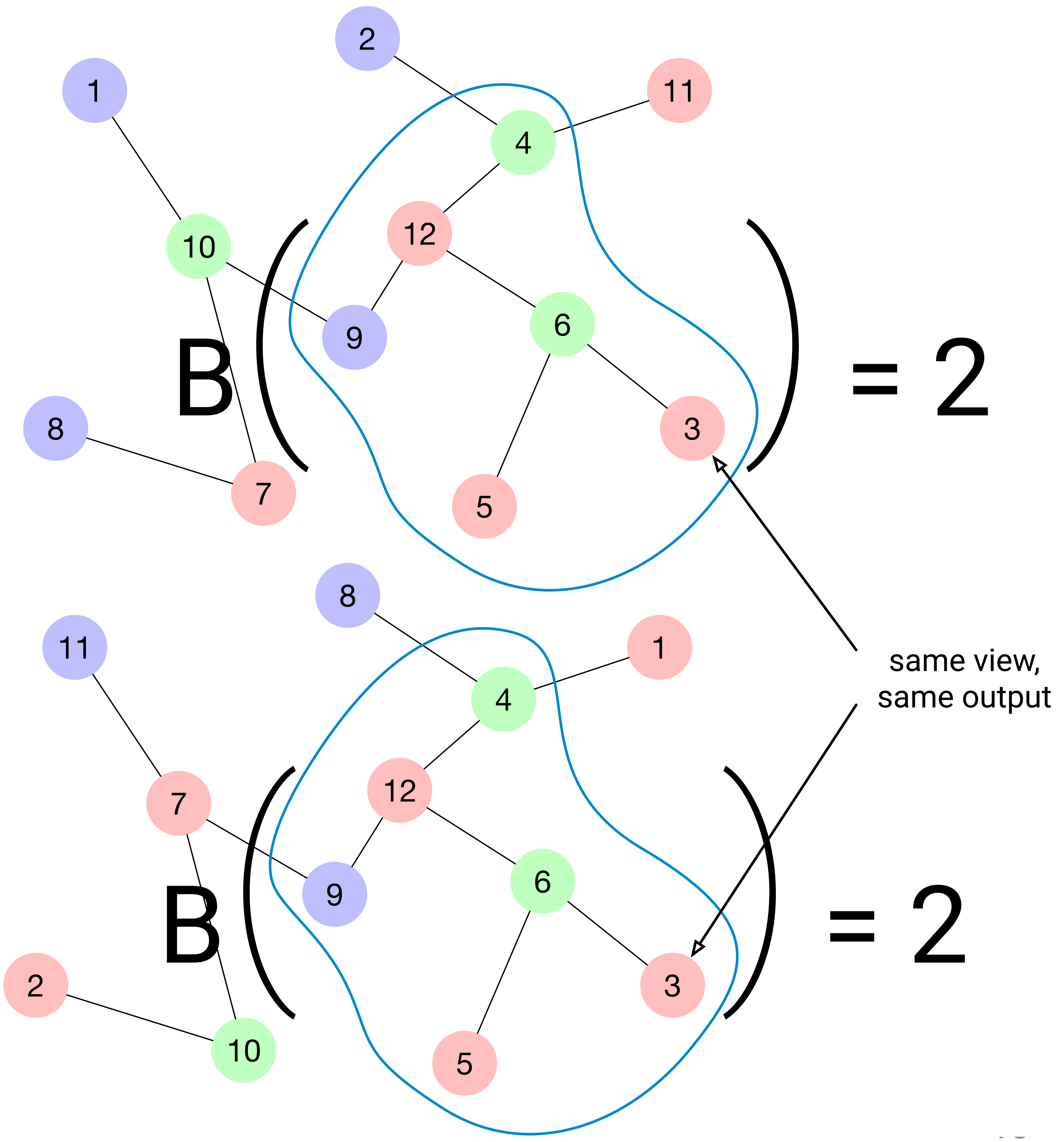

Main technique to prove lower bounds

It can be extended to randomized algorithms:

same radius-T view

⇩

same **probability distribution**

over the outputs

2-coloring

2-coloring

We can solve 2-coloring in **$O(n)$** rounds on paths

2-coloring

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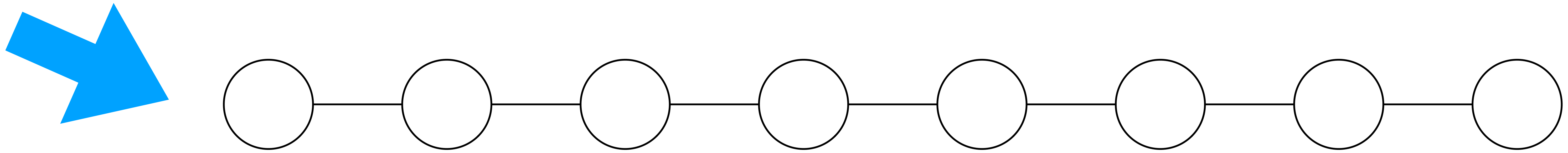

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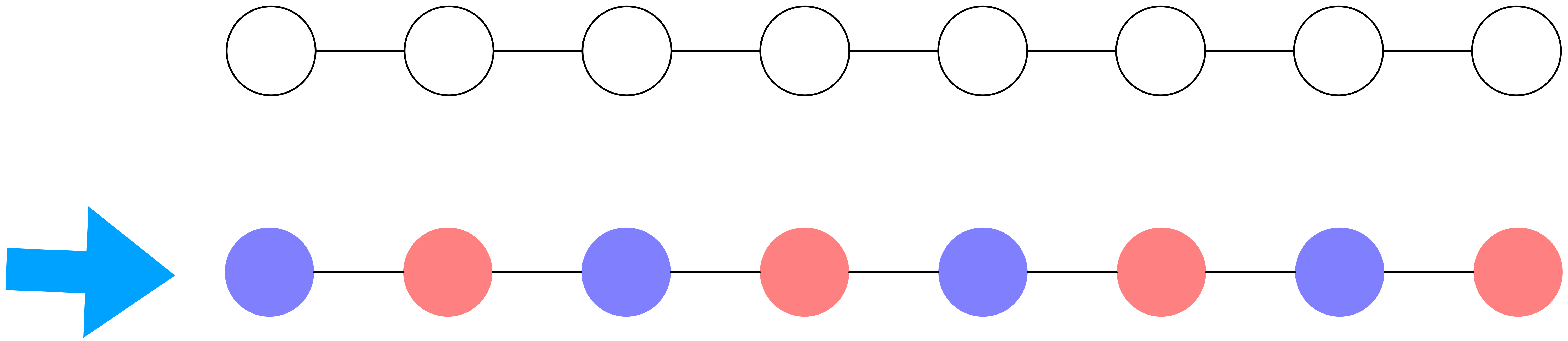

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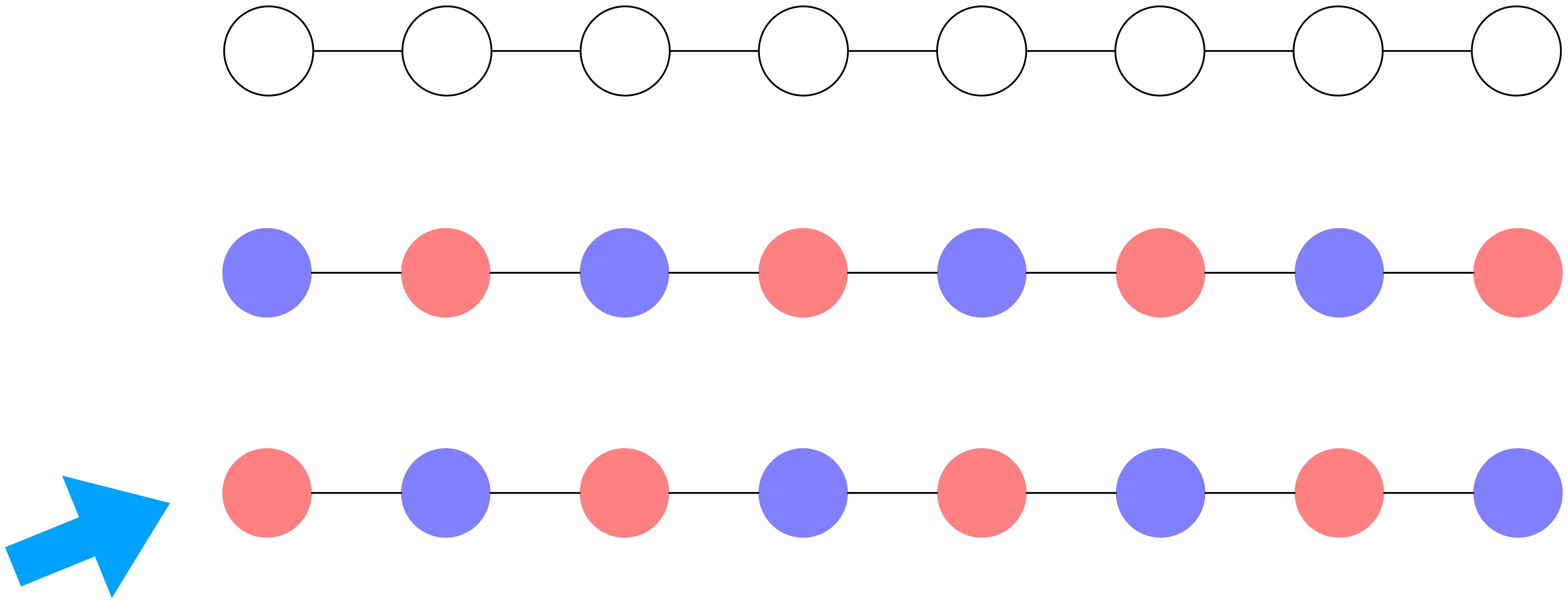

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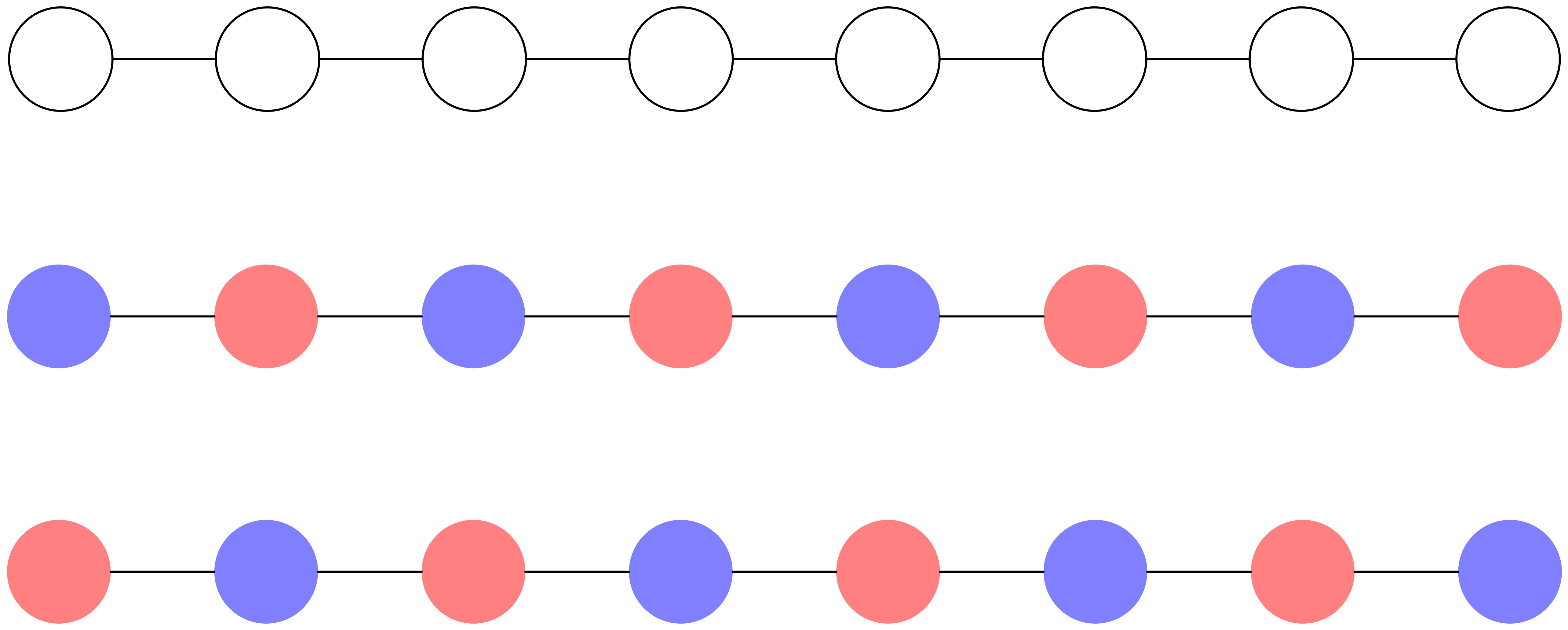

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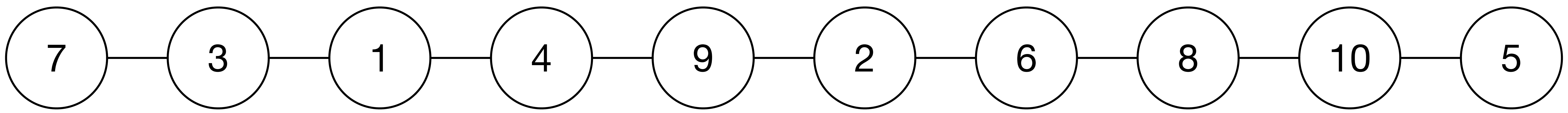

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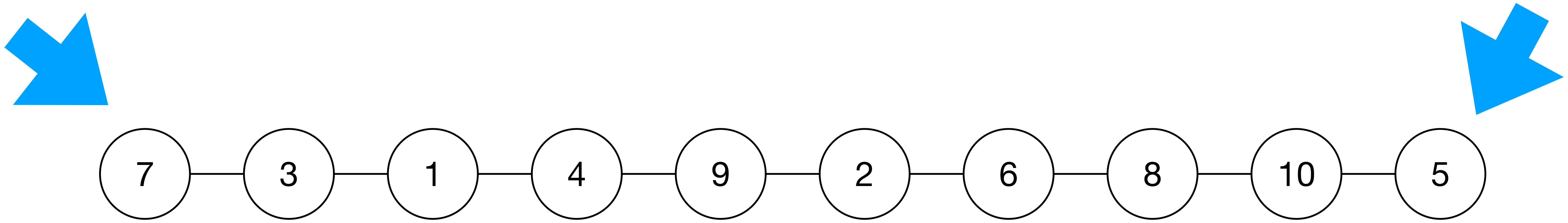

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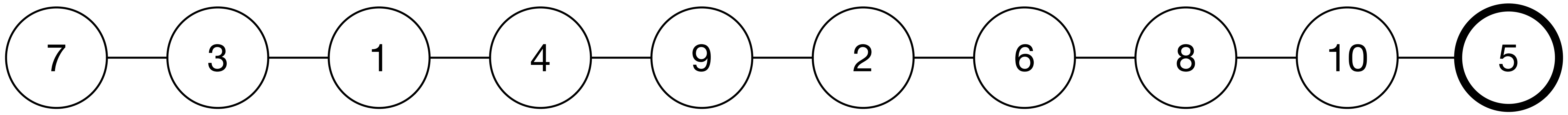

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We can solve 2-coloring in **O(n)** rounds on paths

1. Find minimum among endpoints
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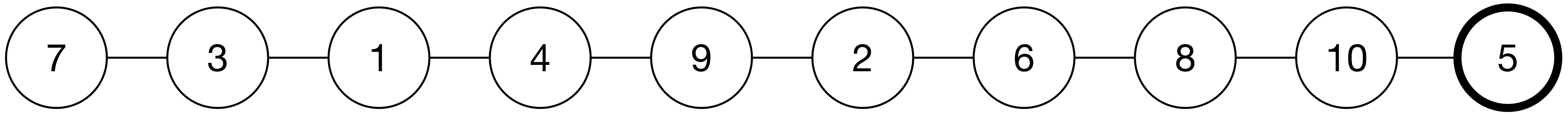

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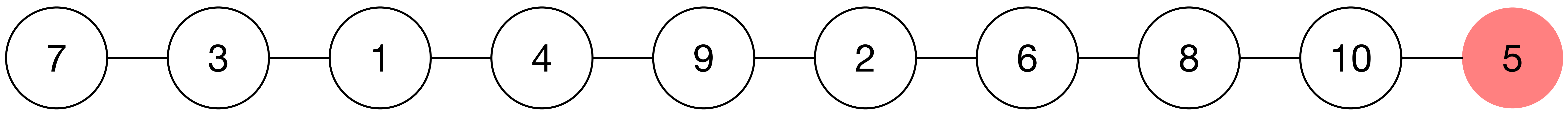

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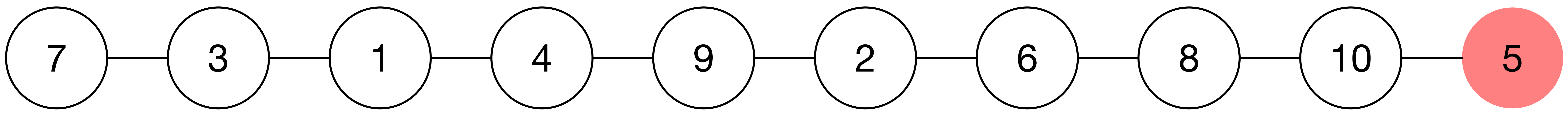

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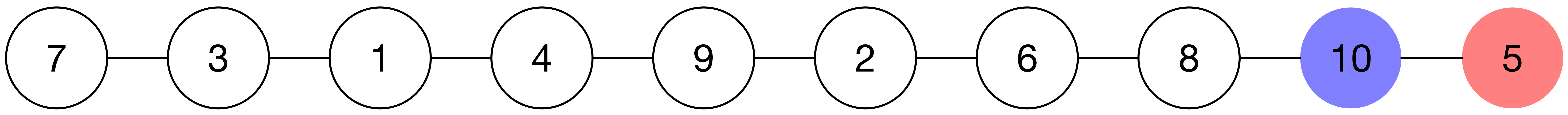

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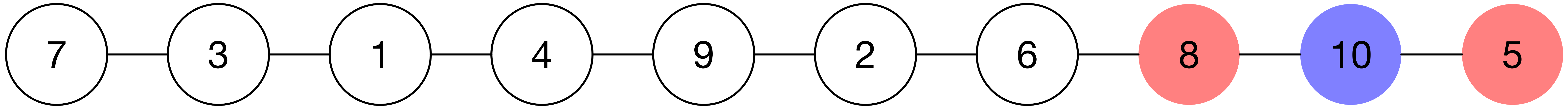

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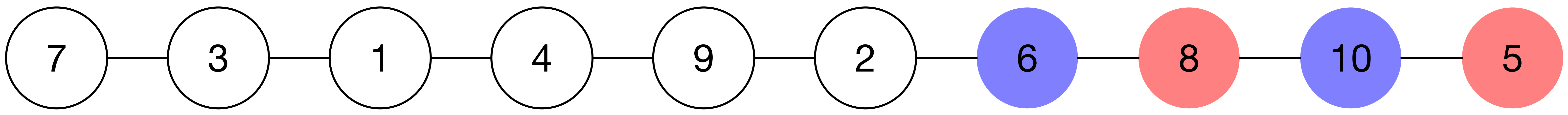

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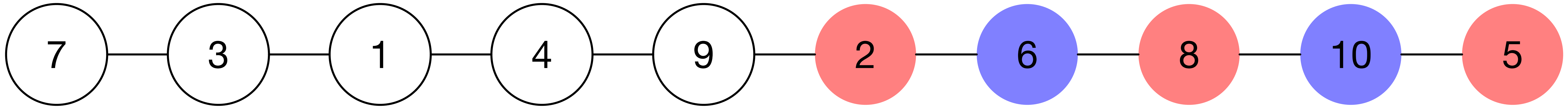

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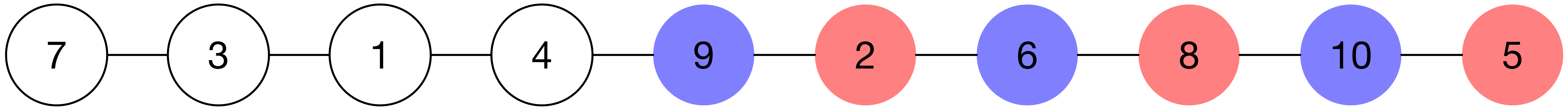

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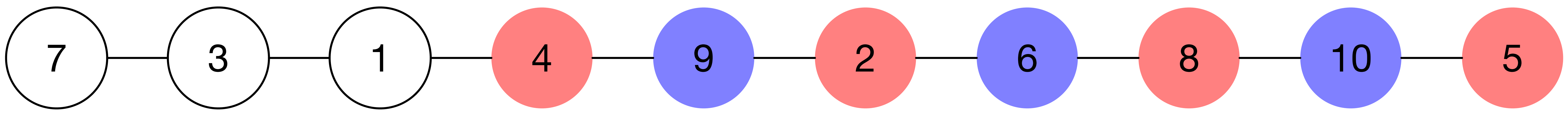

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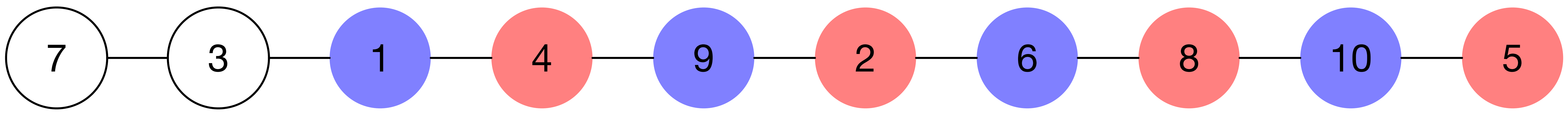

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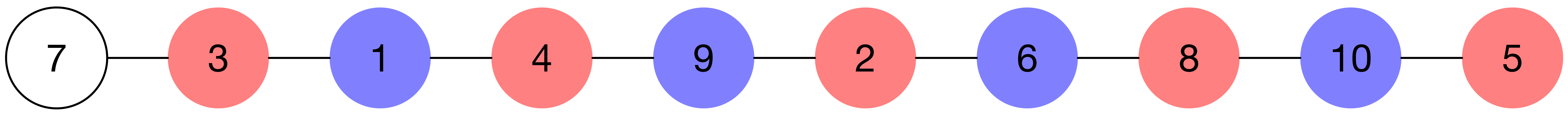

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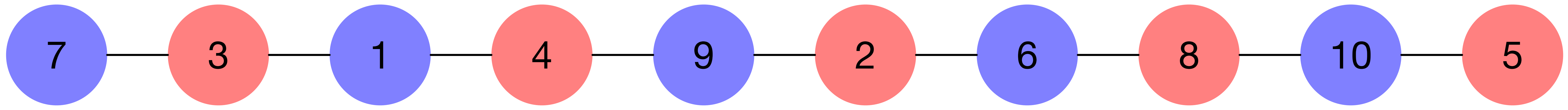

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We can prove that **$\Omega(n)$** rounds are required, even if:

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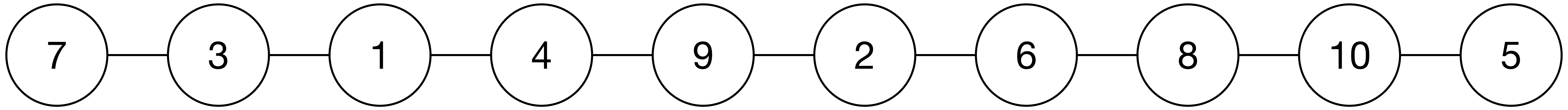

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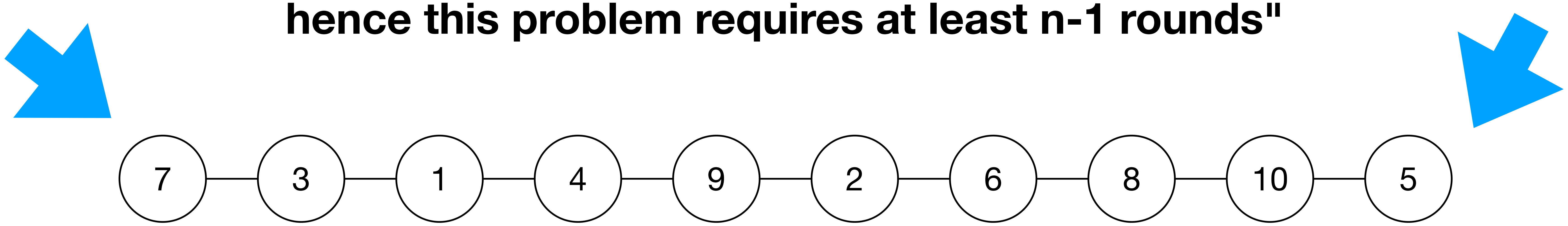

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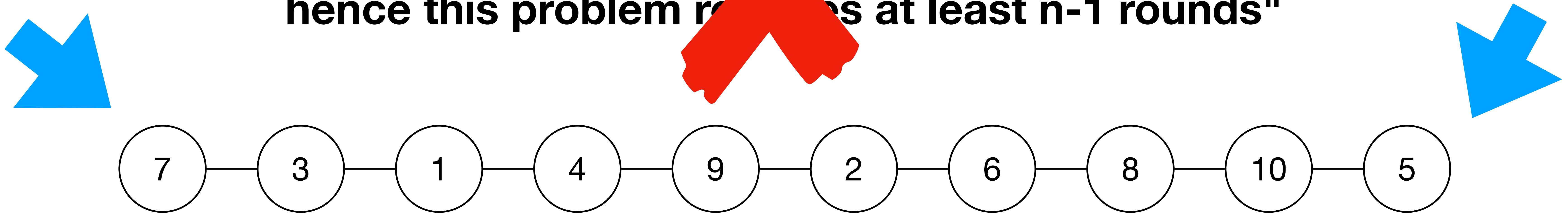

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2-coloring paths can be solved in n/2+1 rounds

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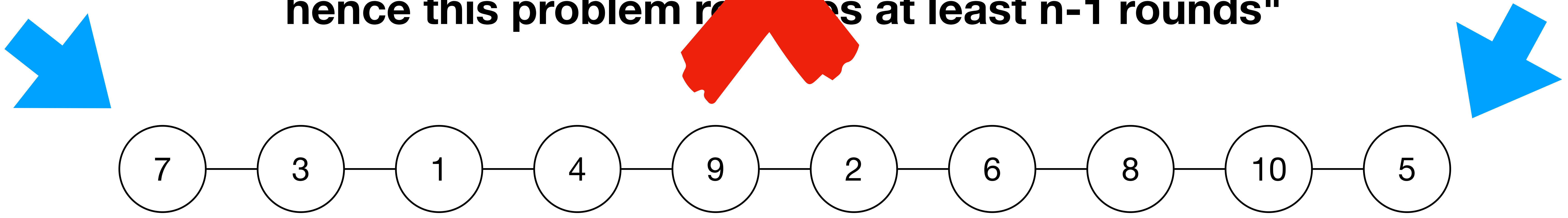

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For any n large enough

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2-coloring lower bound

- Let us prove that **n/5** rounds are not enough, for all (large enough) **n**
- We use the **principle of locality**. We build **two instances** such that:
  - There are two pairs of nodes that have the **same view** in both instances
  - Such nodes **cannot output the same** in both instances

2-coloring lower bound

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For n = 10, **T = 2** is not enough

2-coloring lower bound

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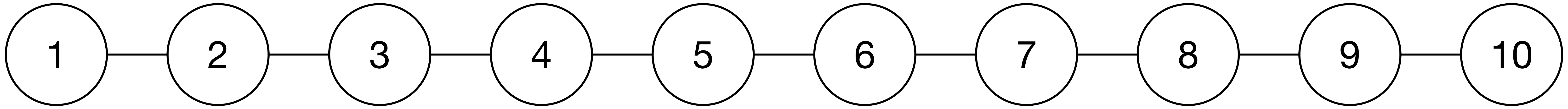

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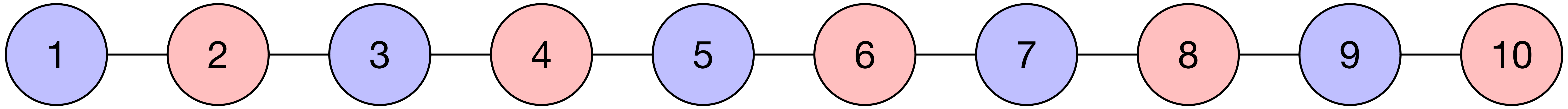

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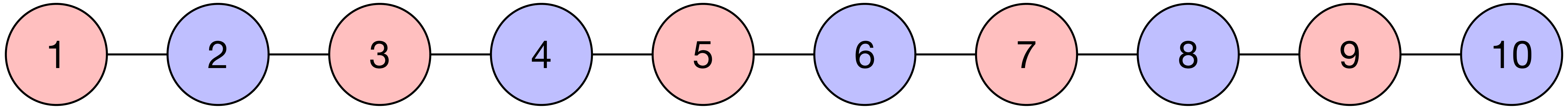

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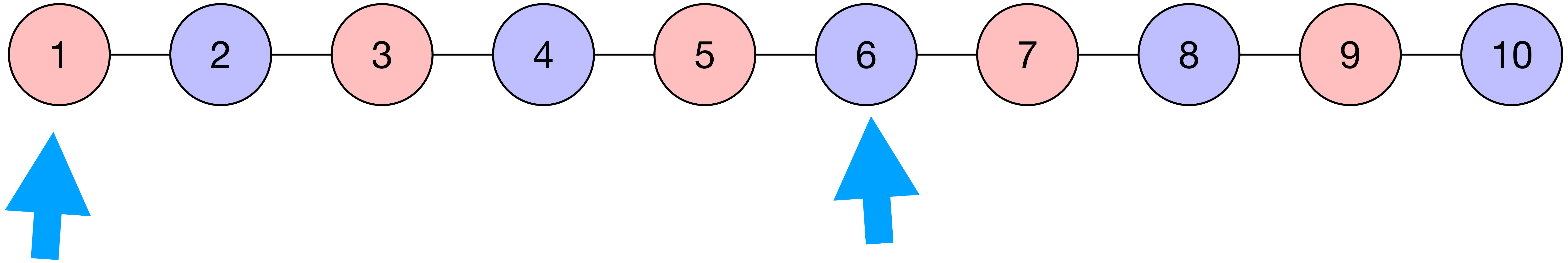

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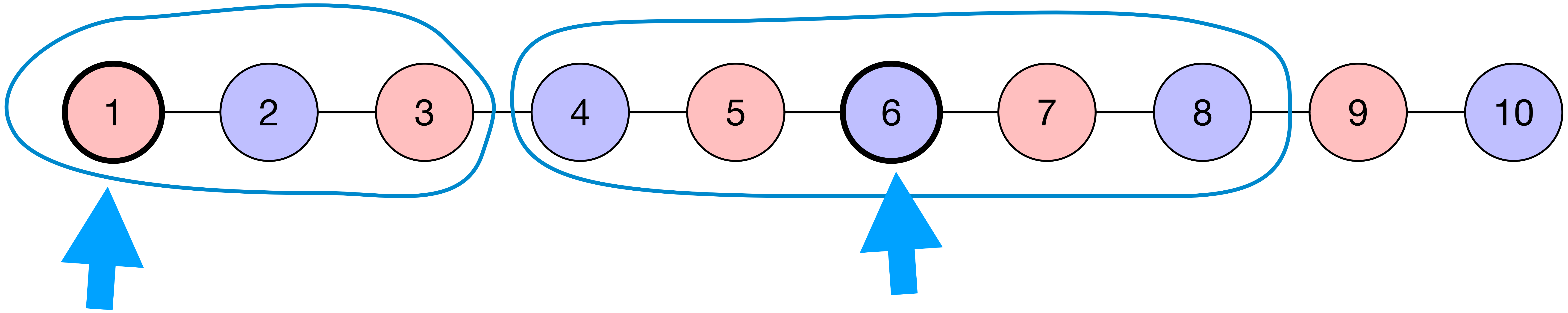

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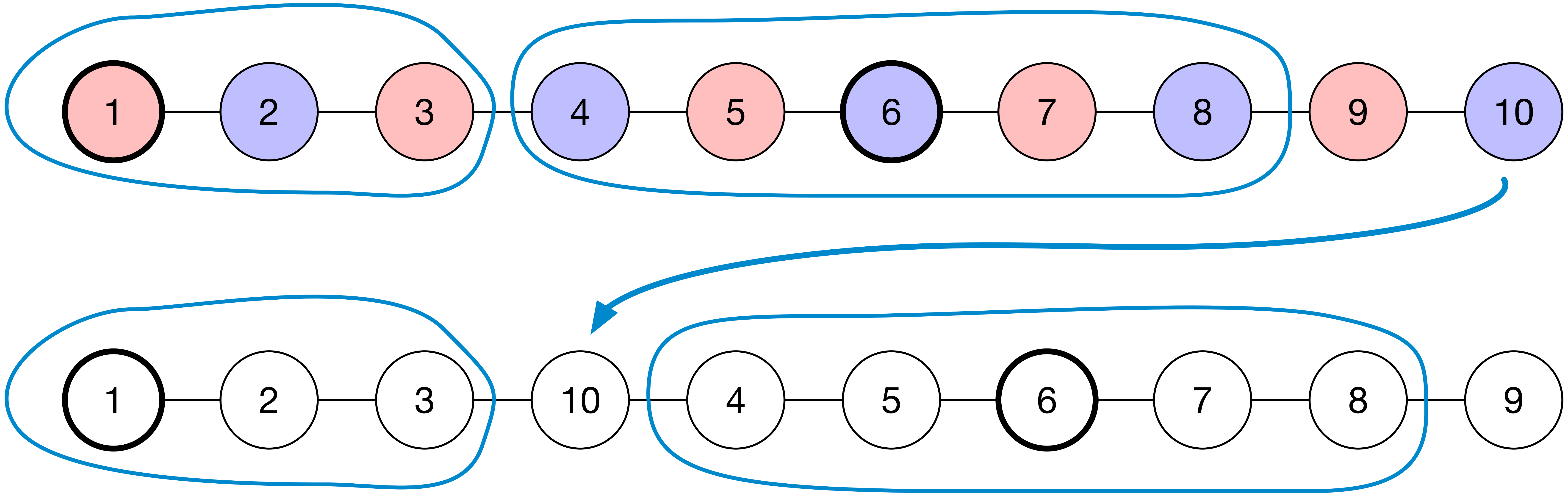

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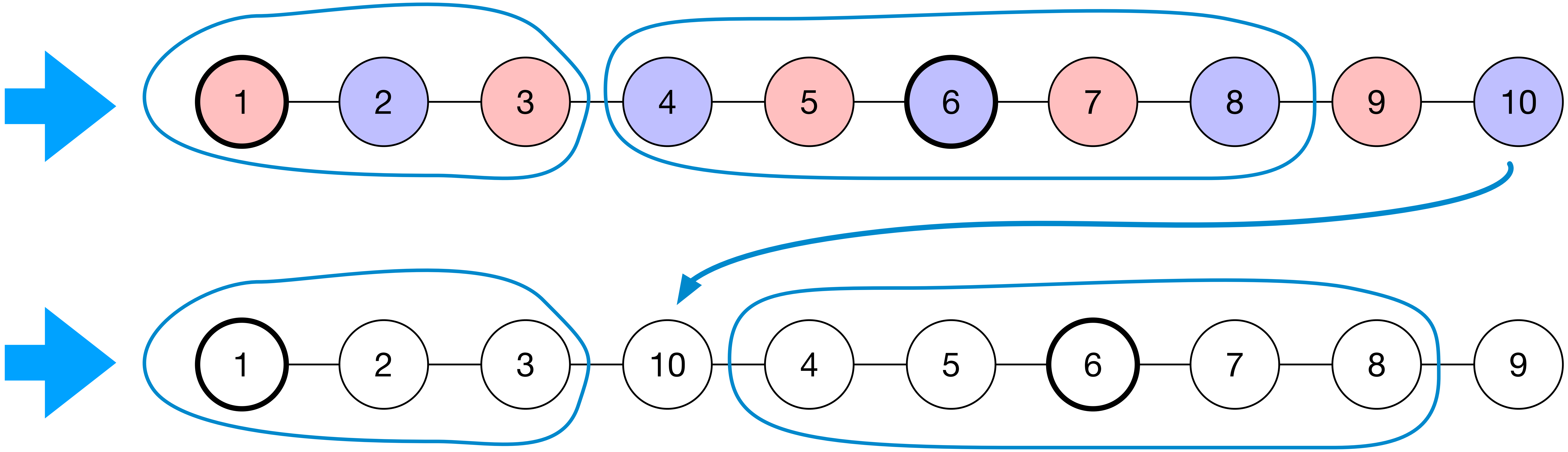

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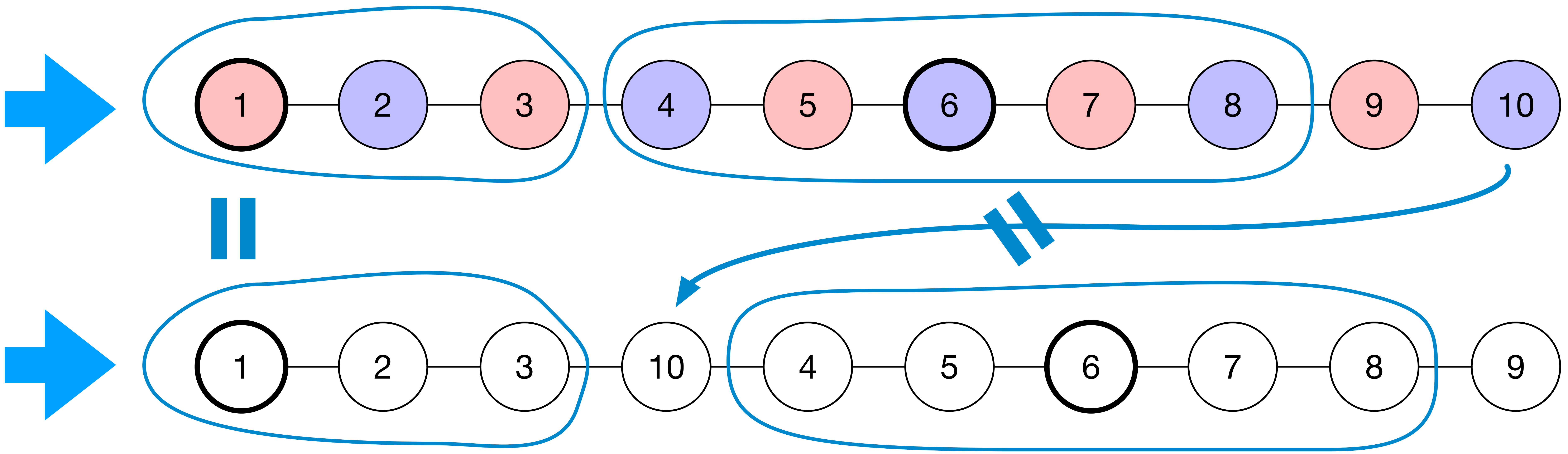

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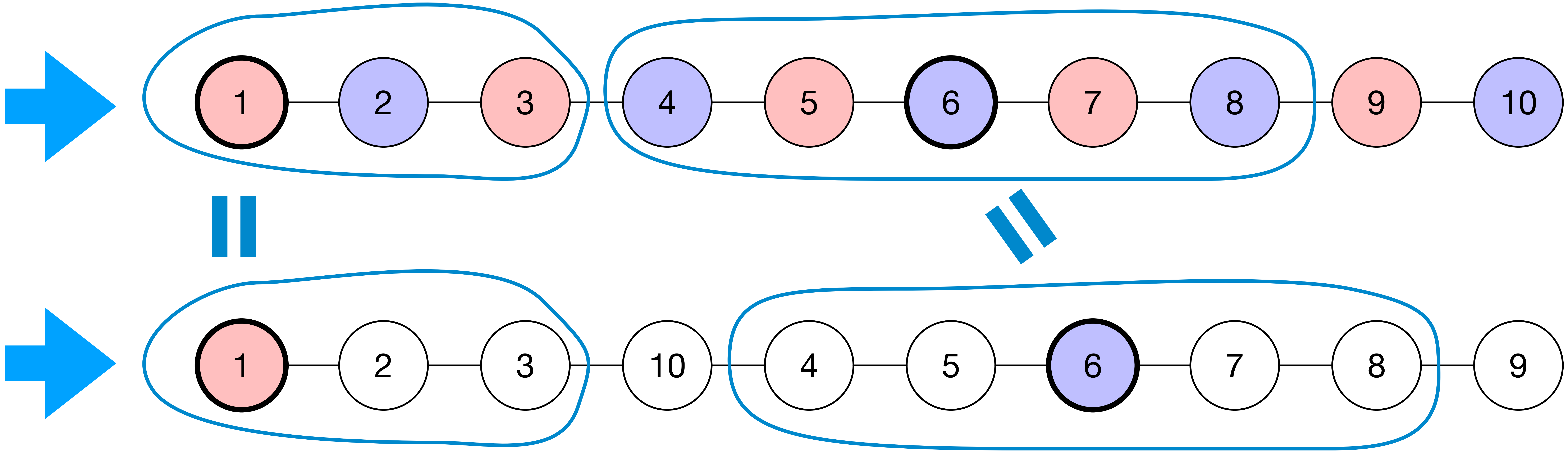

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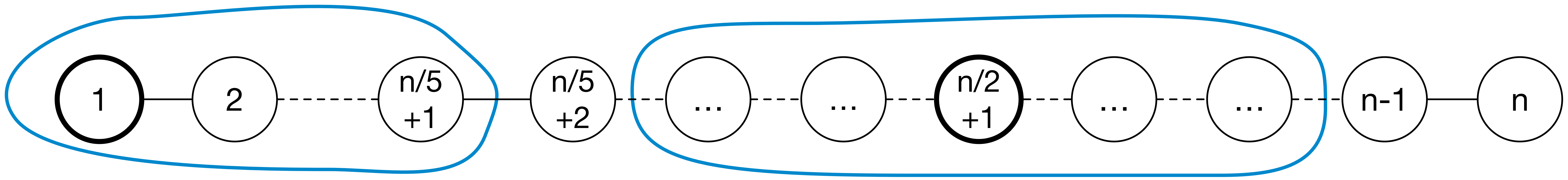

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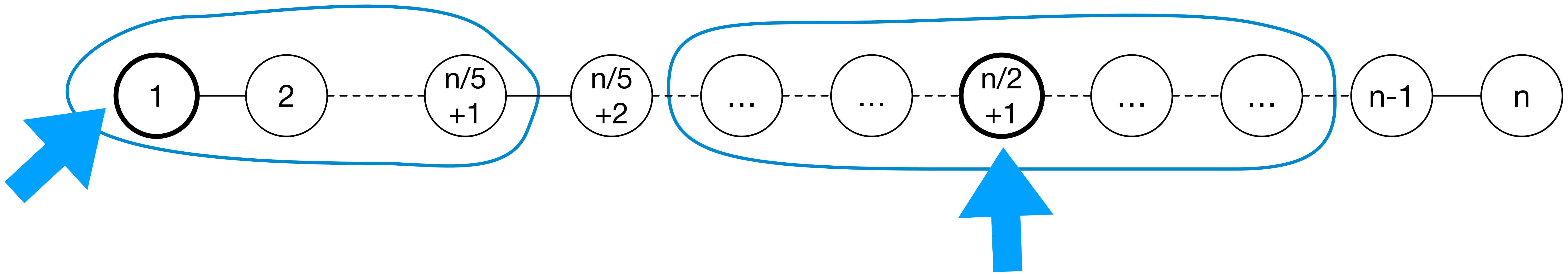
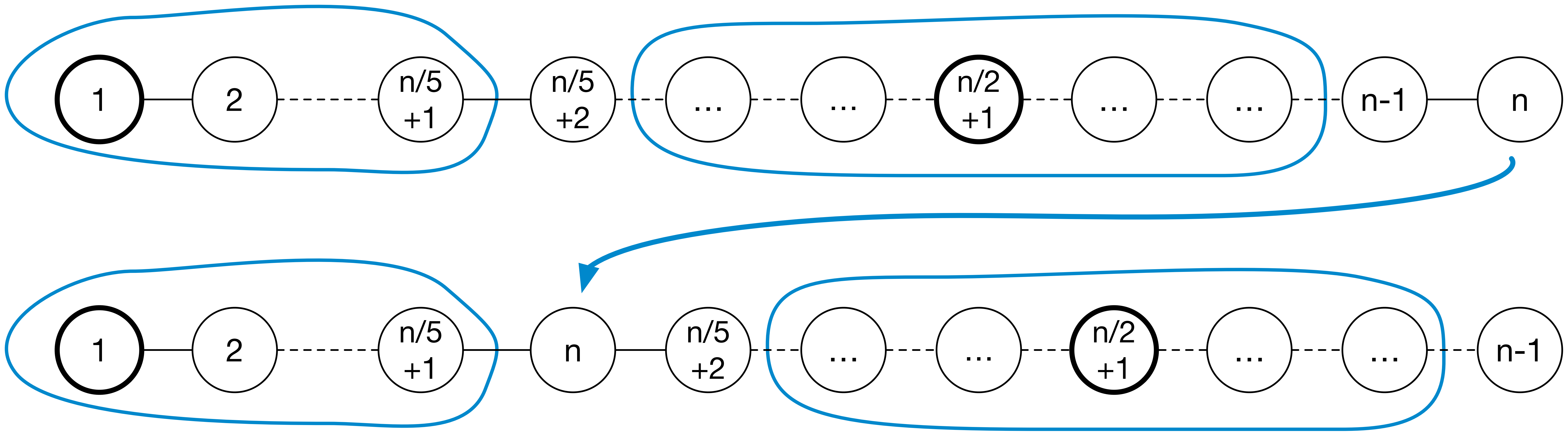


2-coloring lower bound

- Consider the path of length n, where there is an edge between nodes **i** and **i+1**
- Create a new instance, obtained by **removing** the edges **{n-1,n}** and **{n/5+1,n/5+2}**, and **adding** the edges **{n/5+1,n}** and **{n,n/5+2}**



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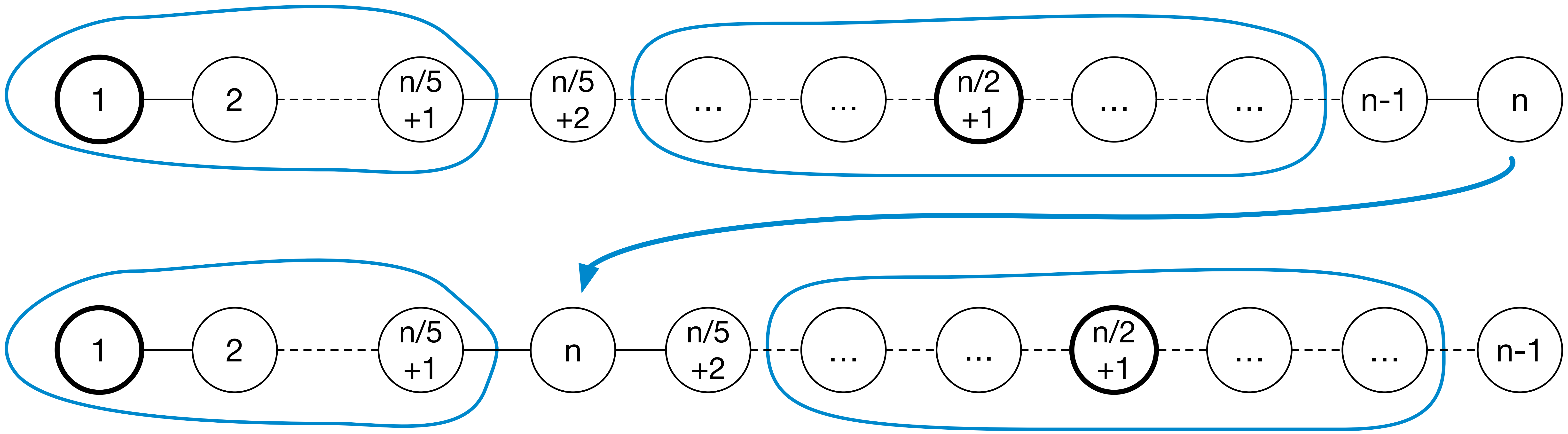
- For large enough n, nodes **1** and **(n/2+1)** have the **same radius-n/5 view**, hence they must output the same in both instances, but this is **wrong** (the distances of these nodes in the two instances have different parity)



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- Main ingredient:

  same radius-T neighborhood

  ⇩

  same **probability distribution**

  over the outputs

Coloring trees

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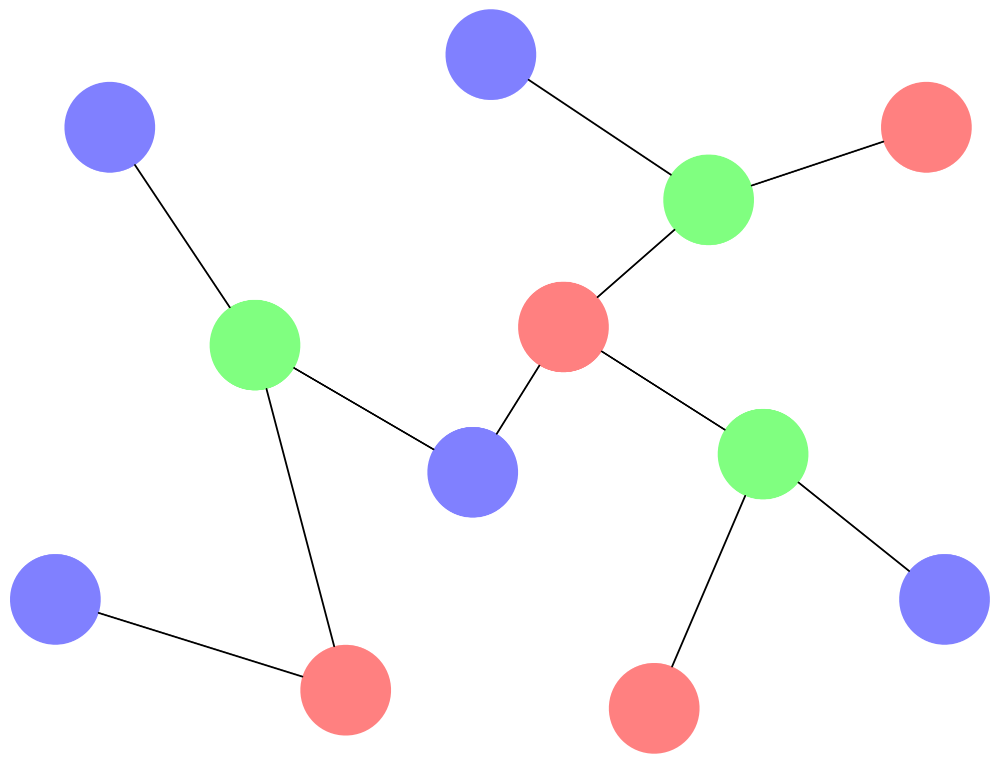

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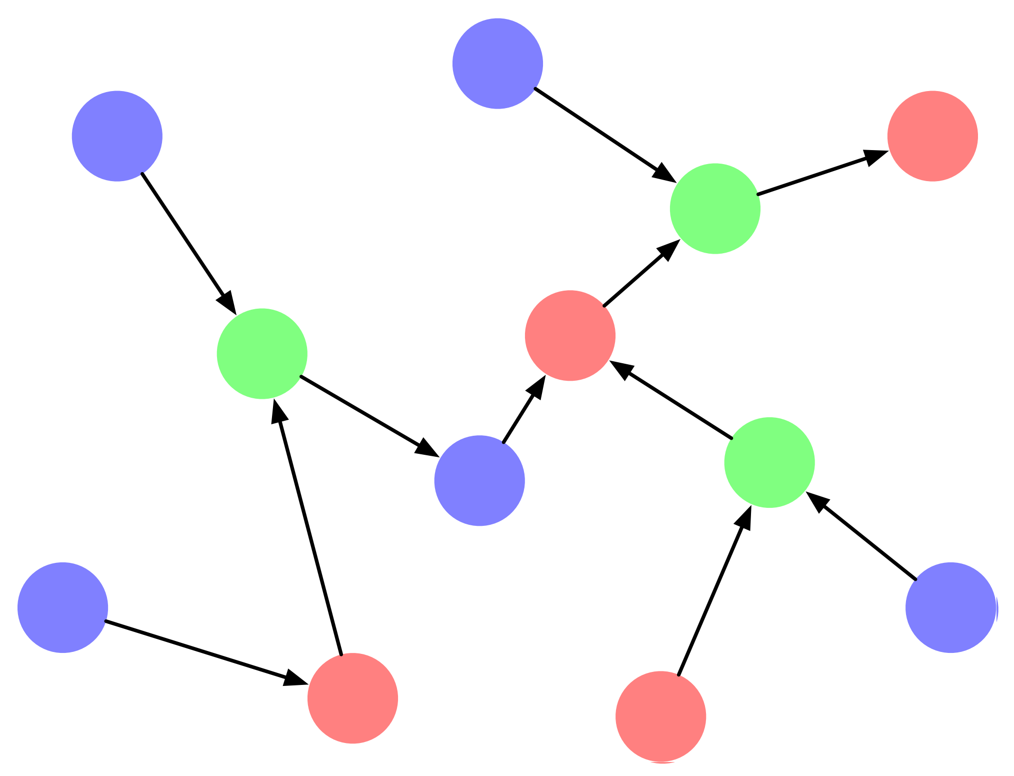

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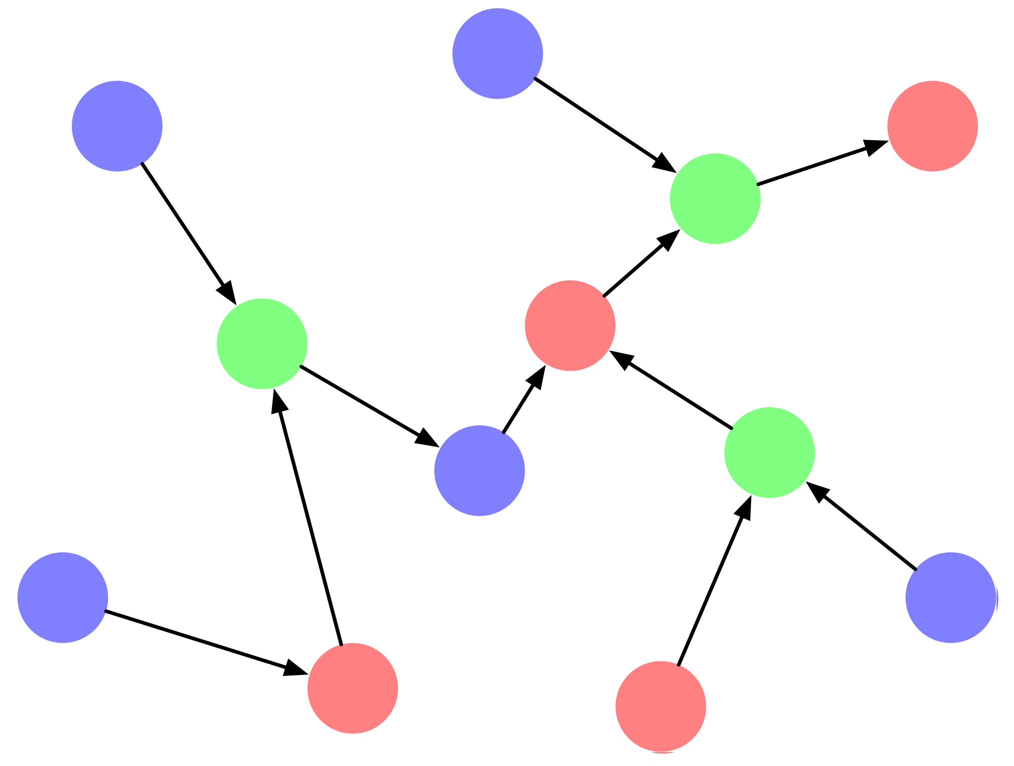

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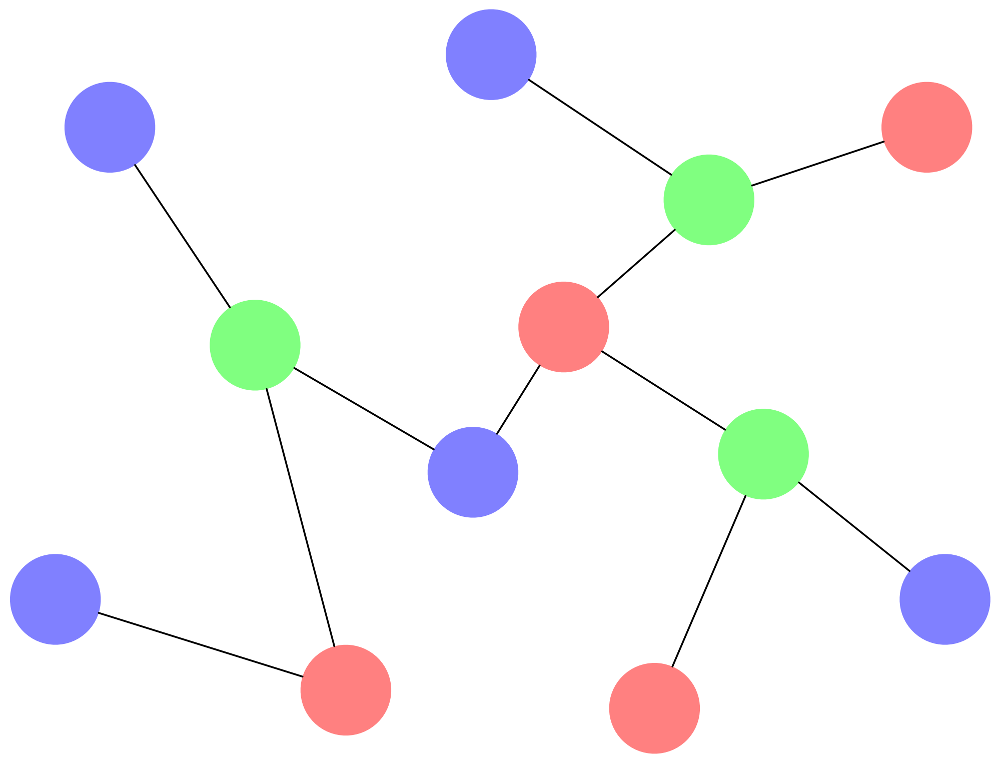
Coloring trees



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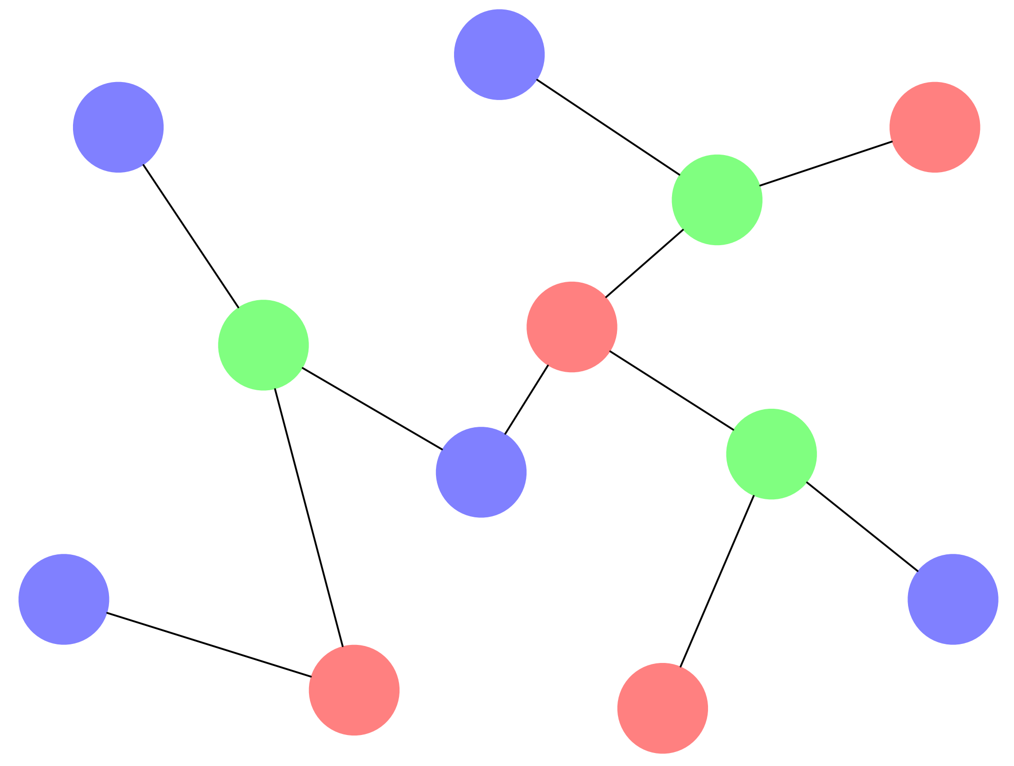

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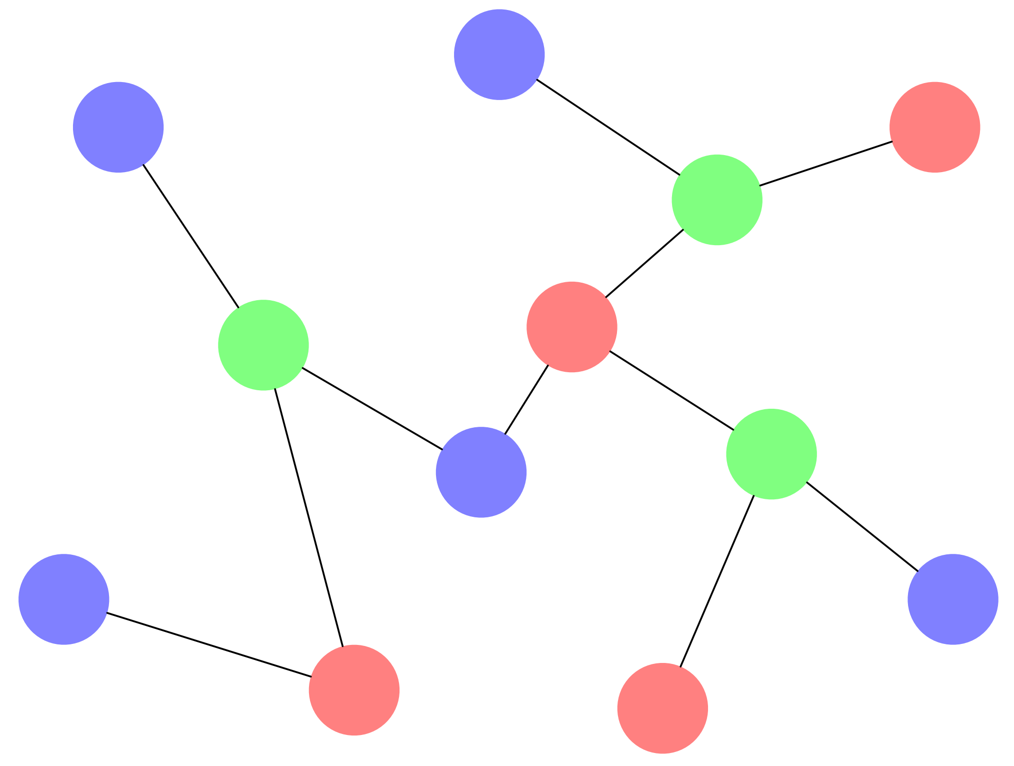

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⇒ **3**-coloring trees of (large enough) constant degree requires **Ω(log n)** rounds

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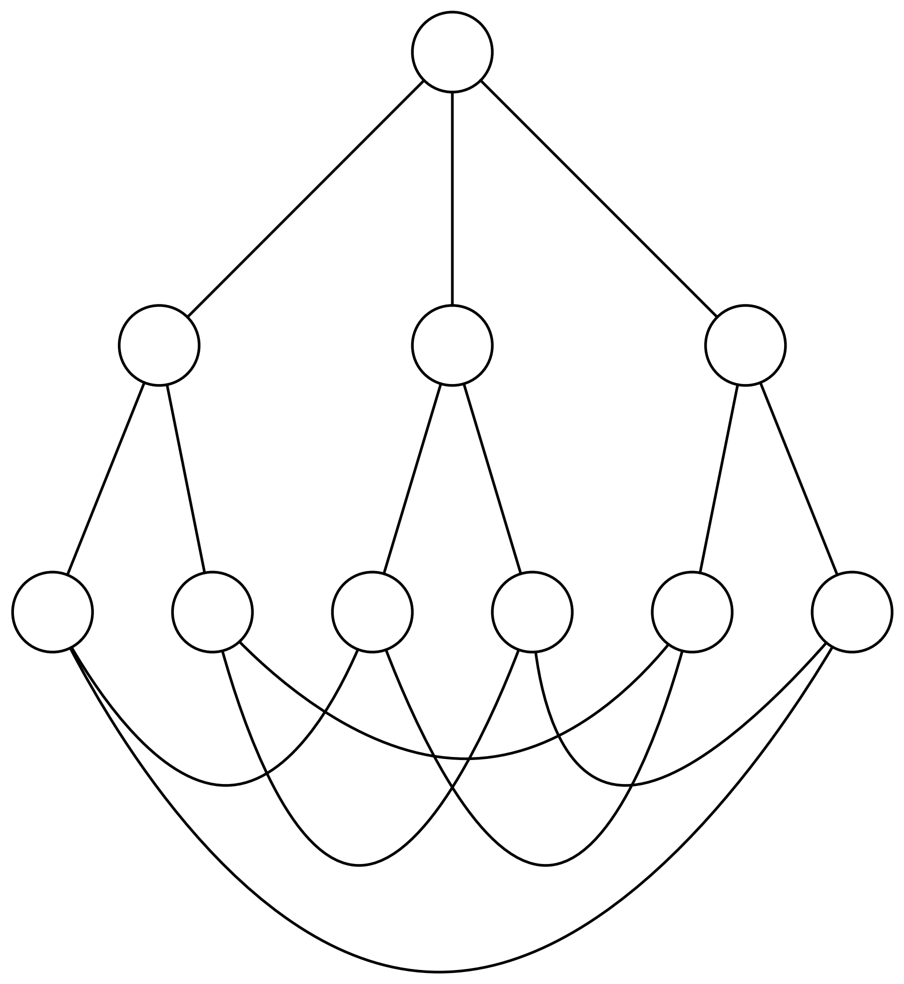

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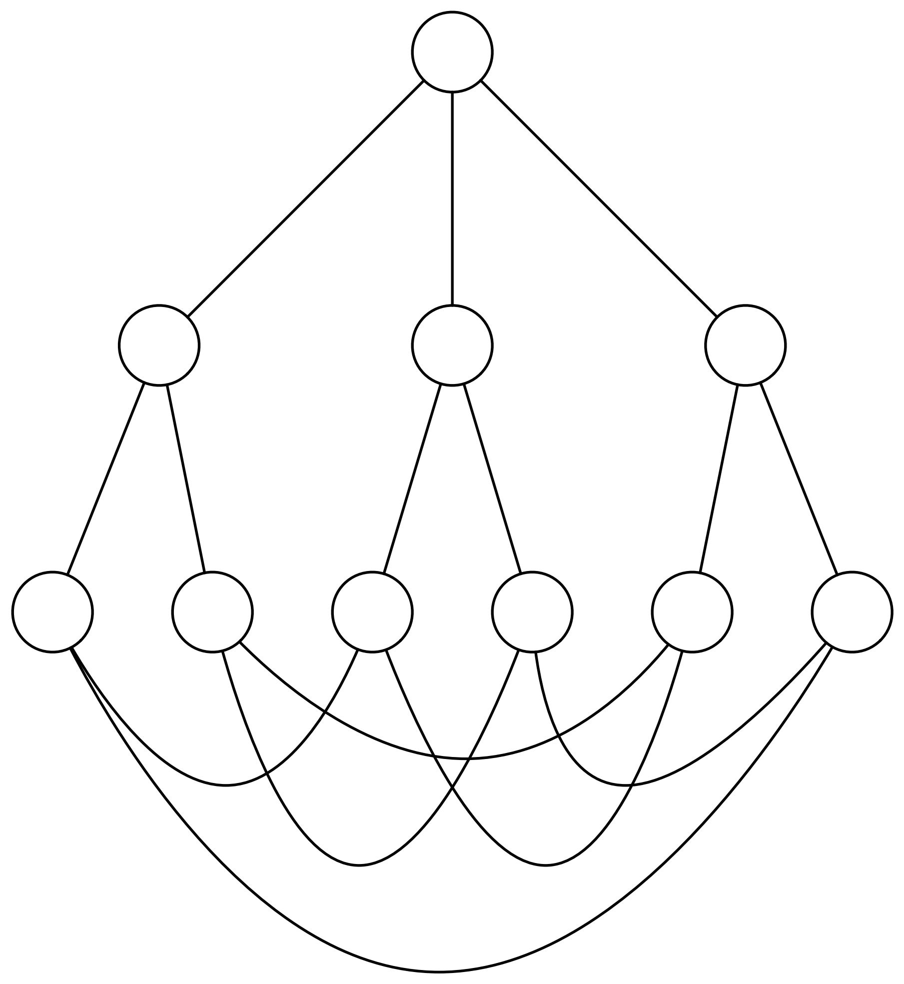

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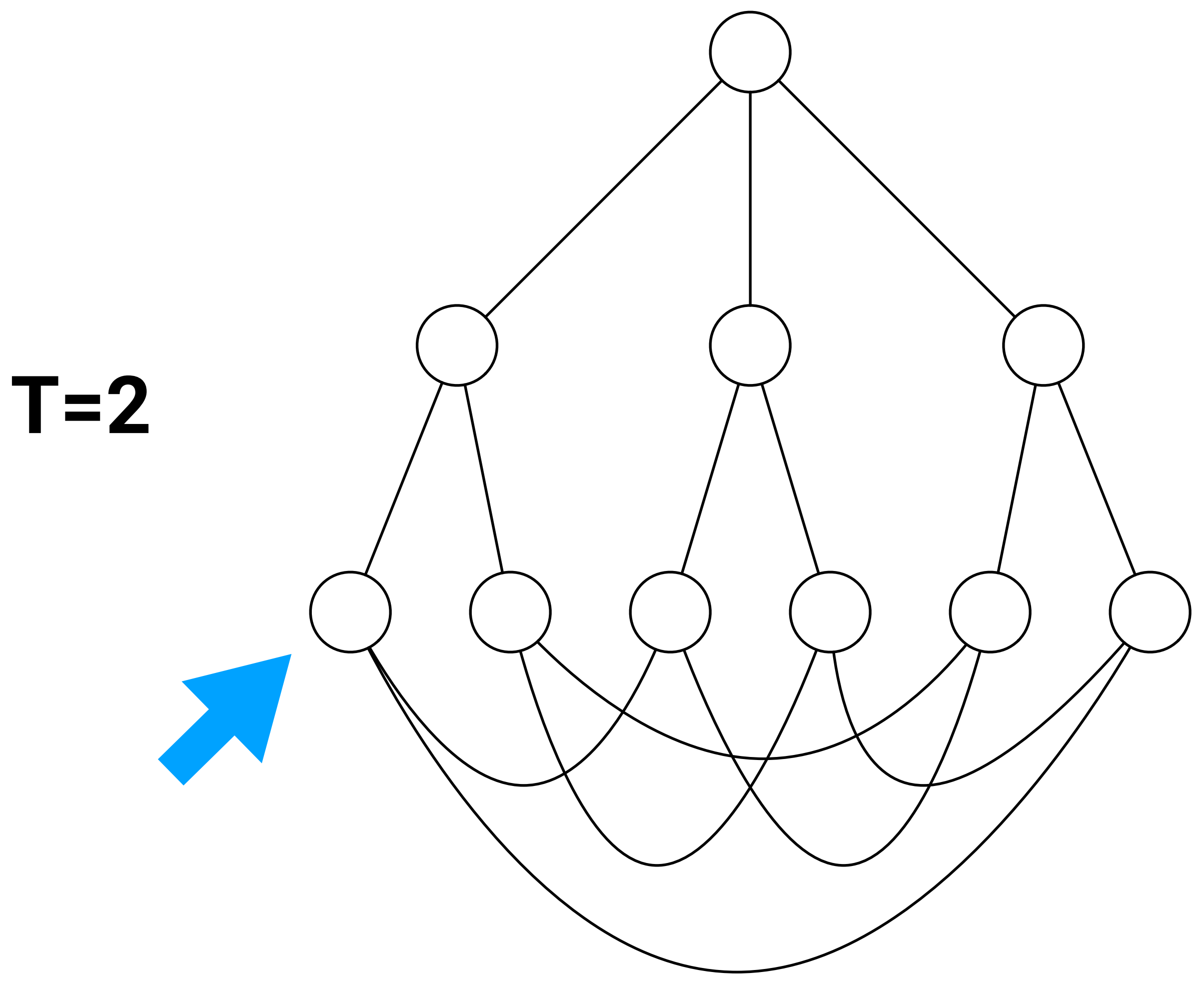

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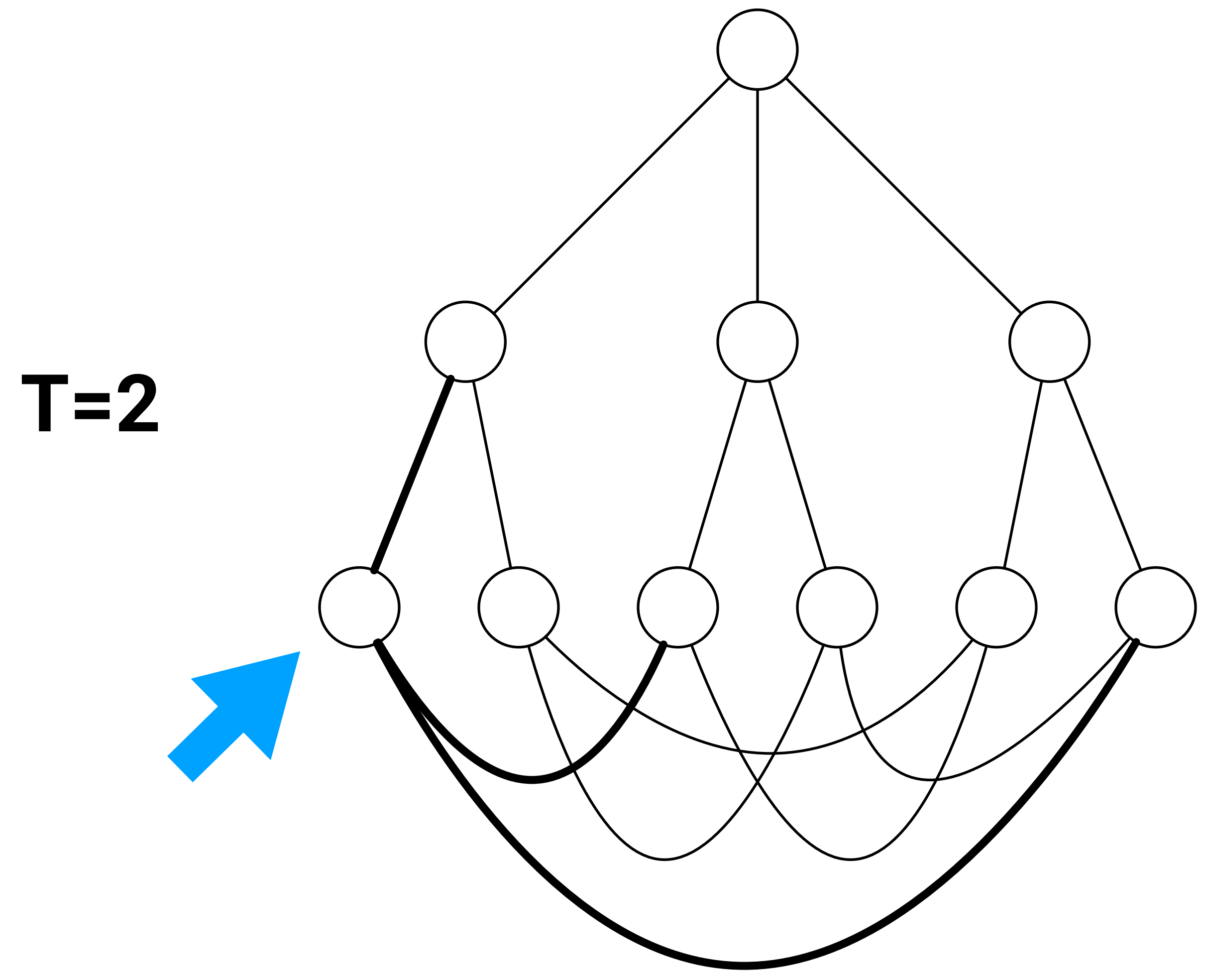

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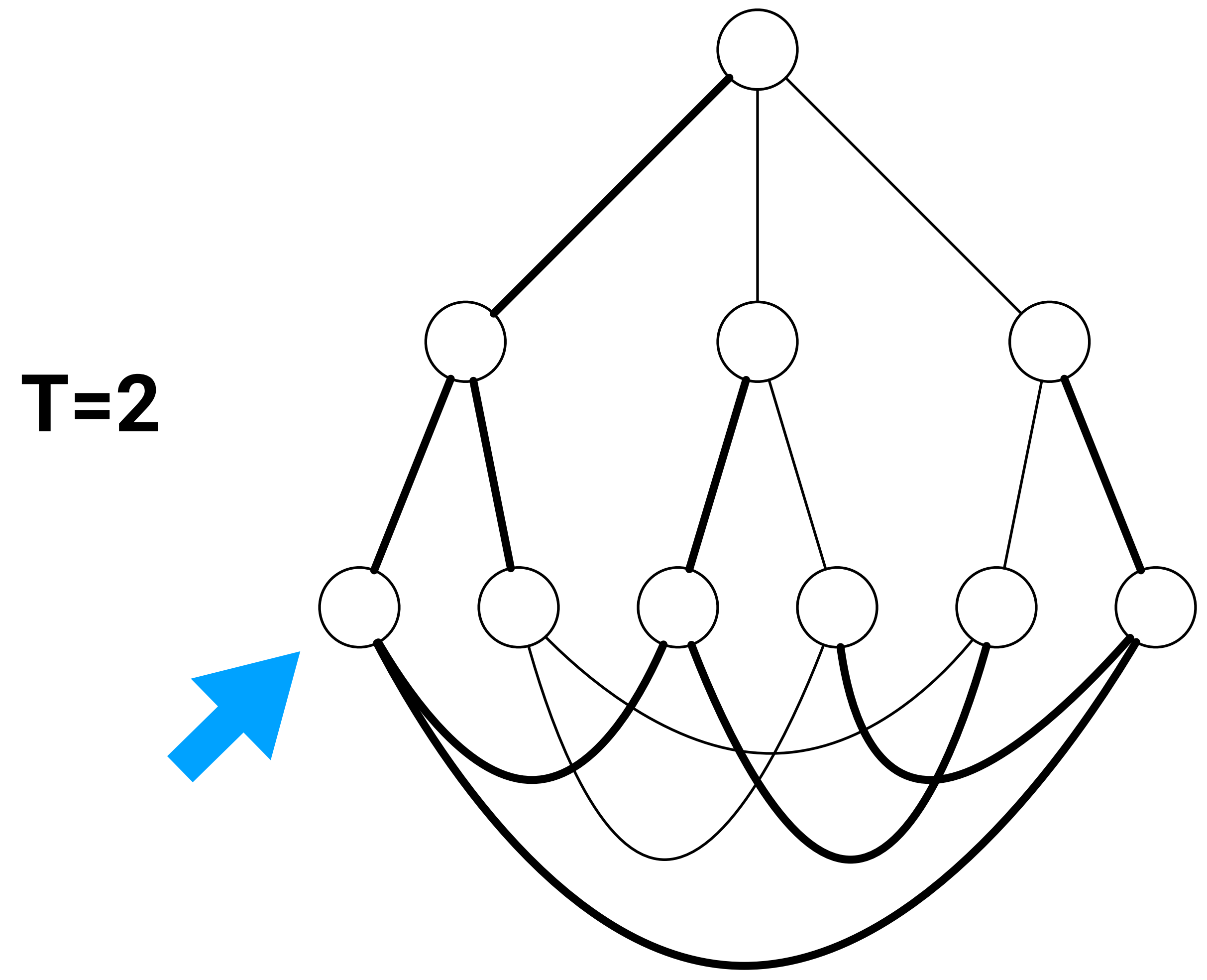

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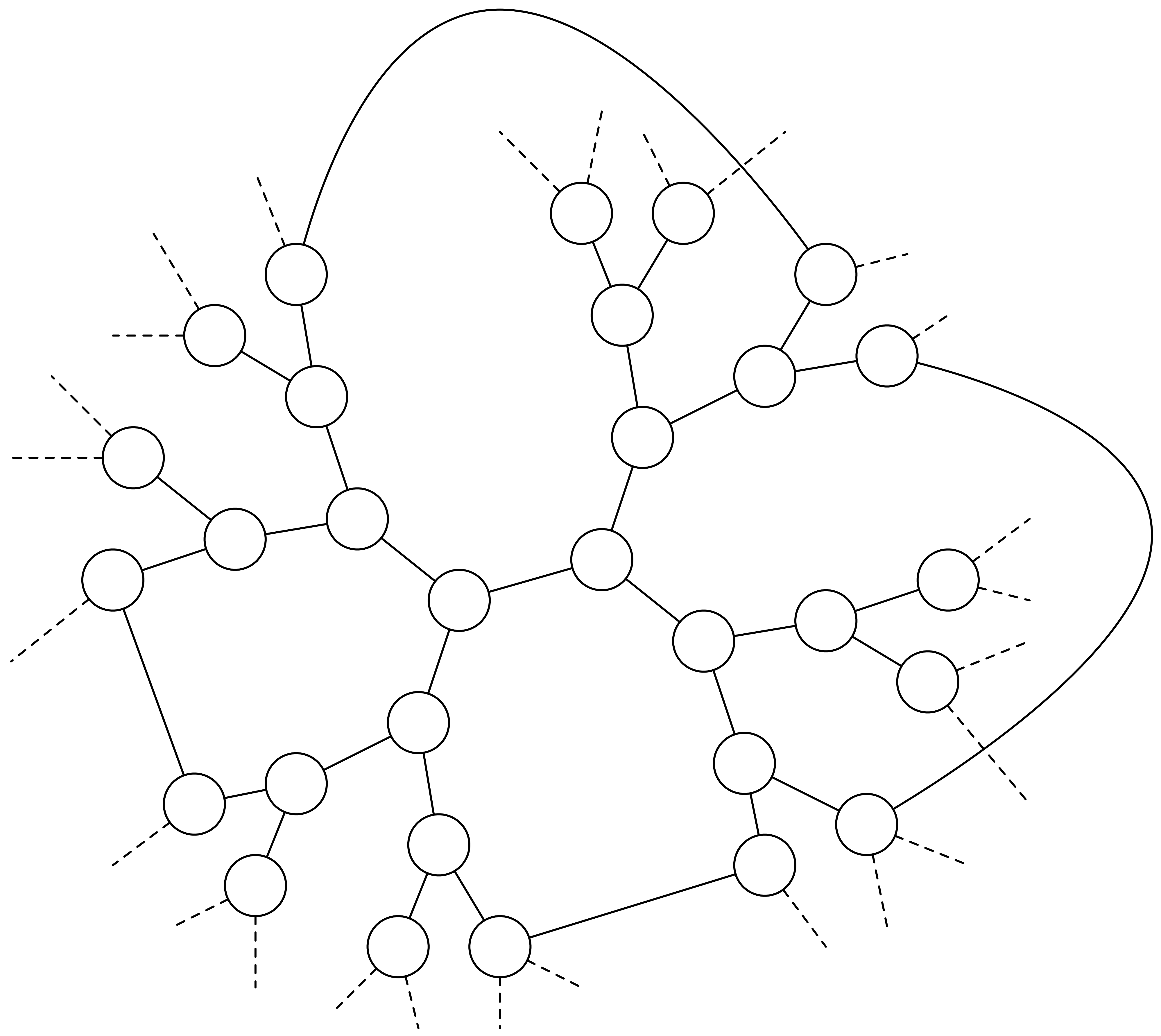
  - We now prove that such failure implies that **A** must also **fail** on some specific **tree**

Coloring trees lower bound



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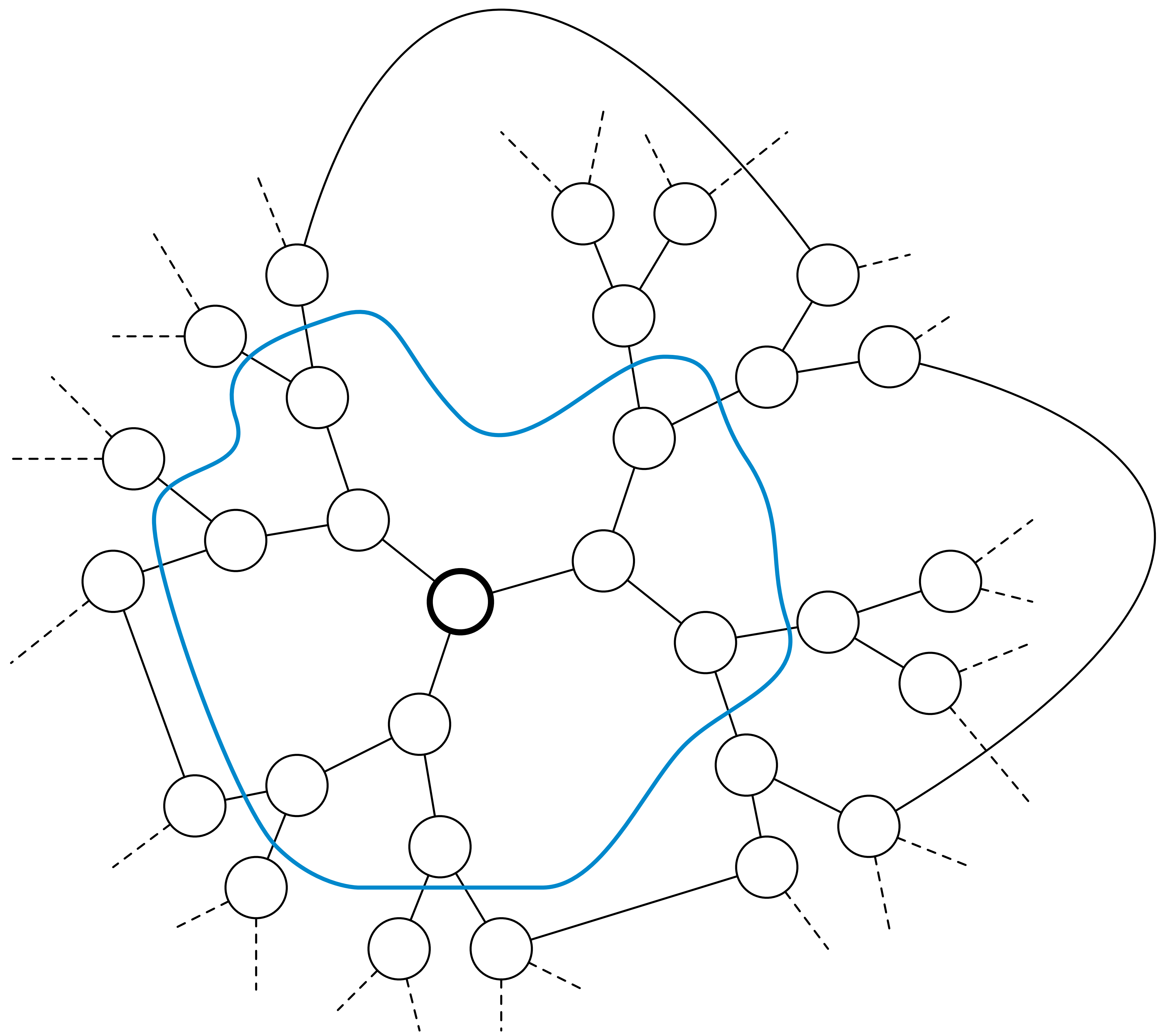

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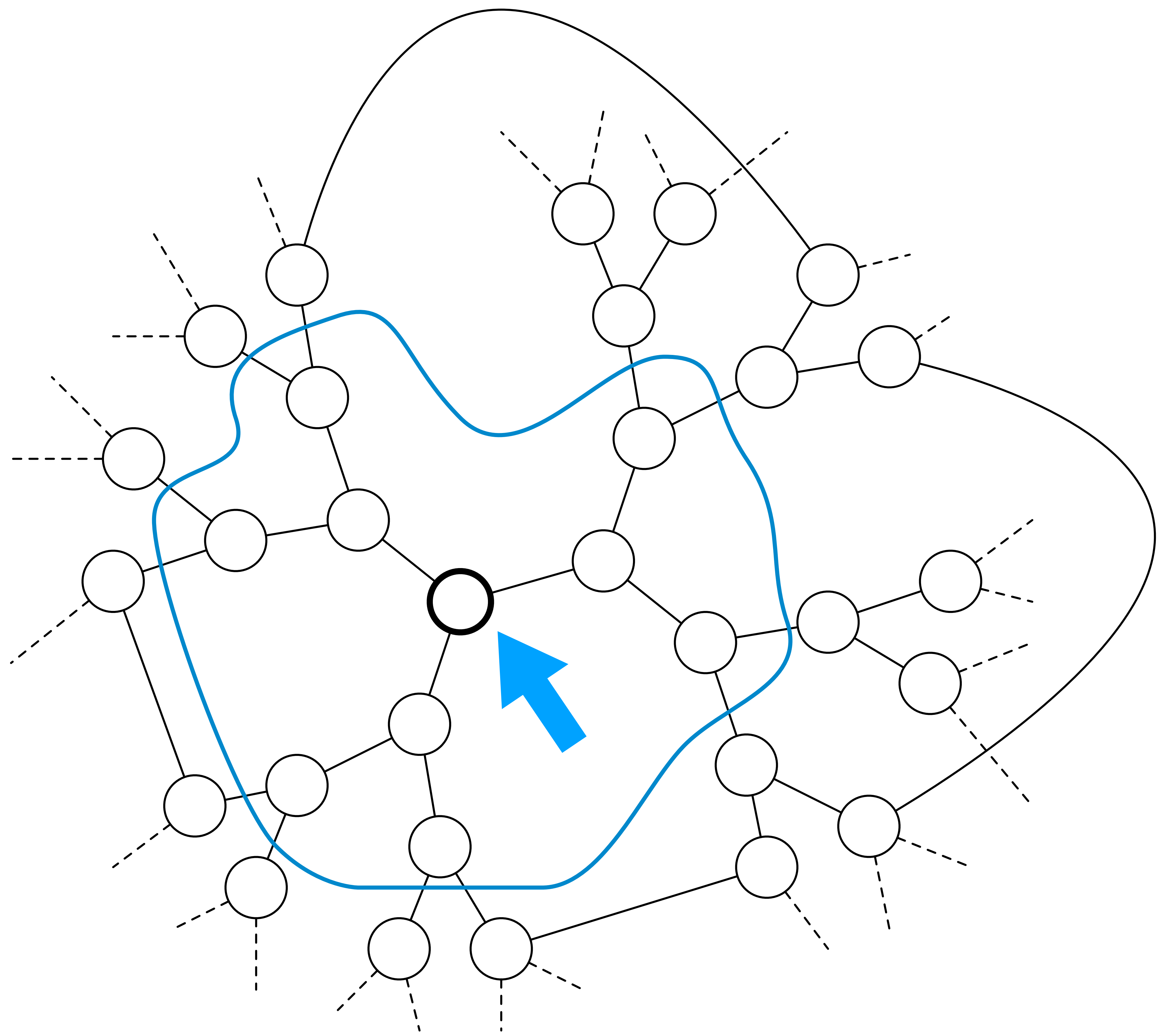

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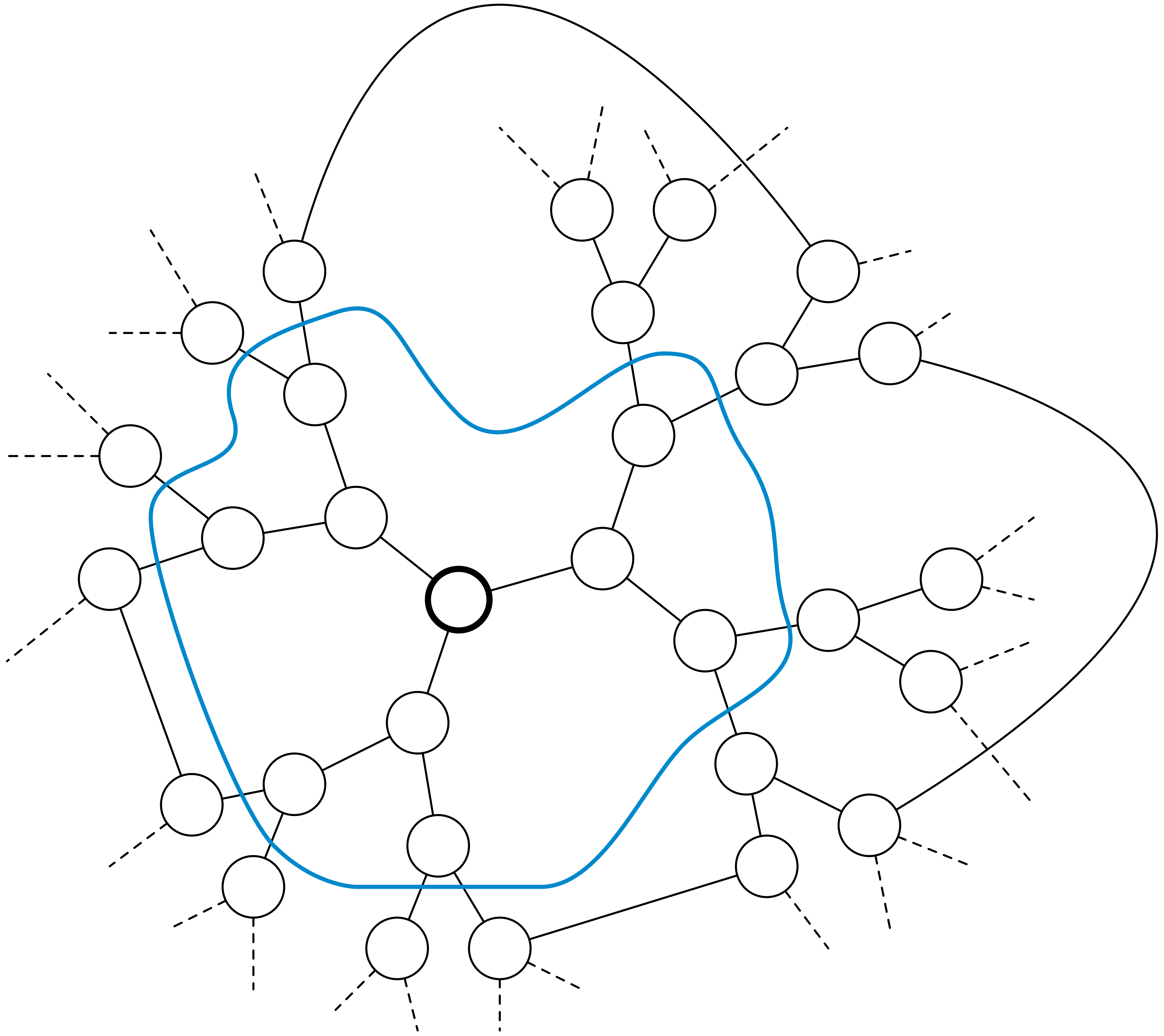

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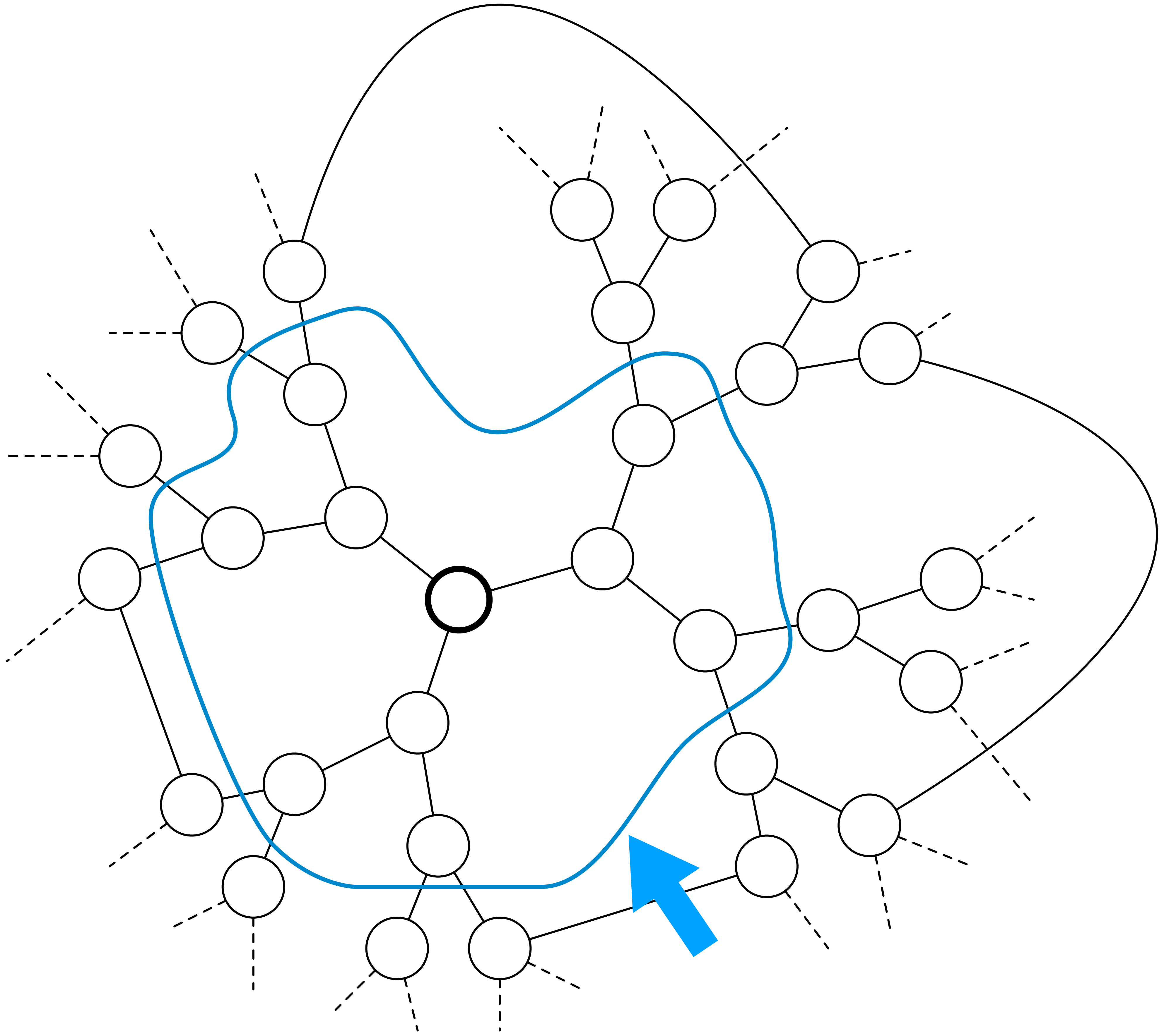

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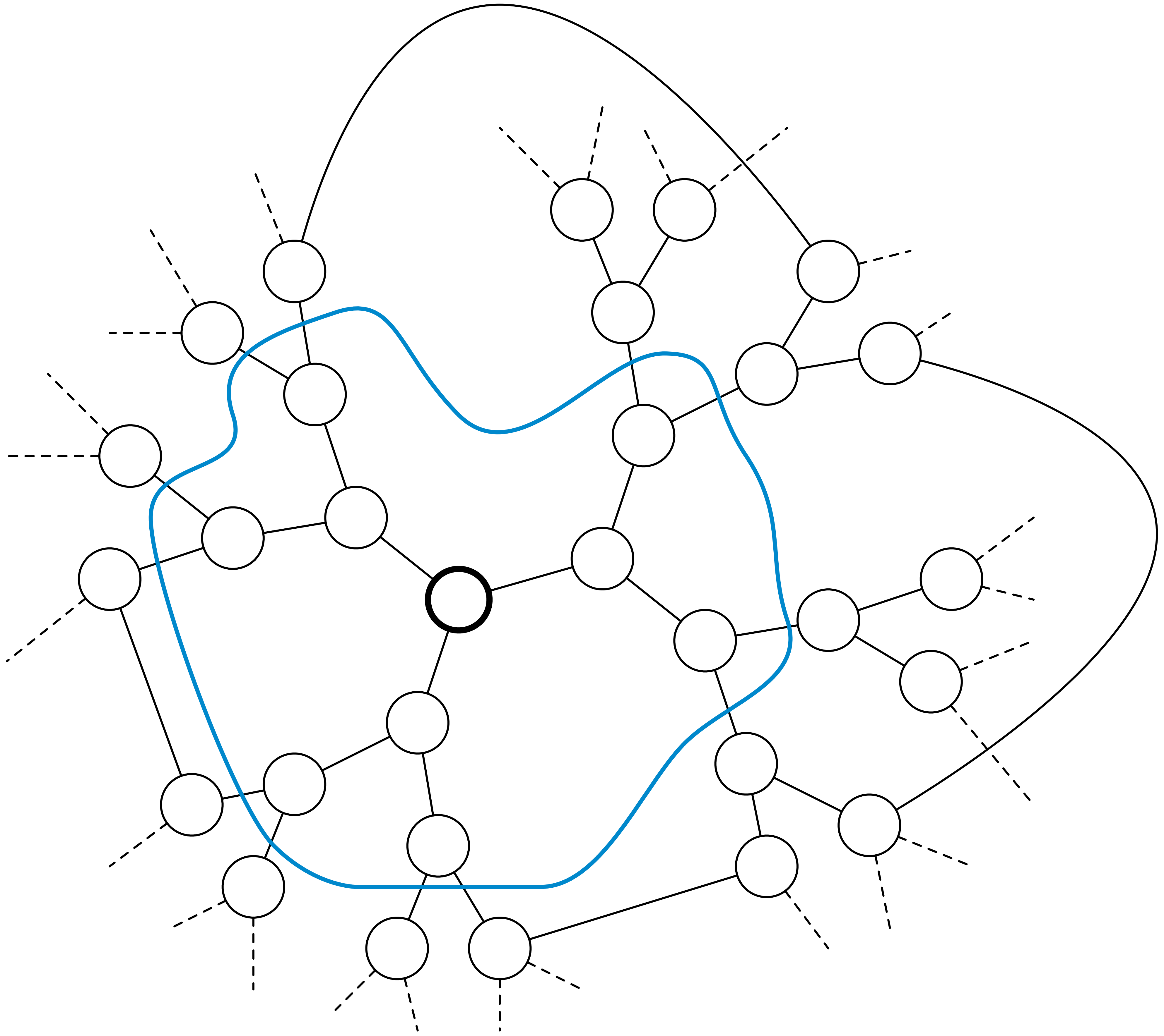

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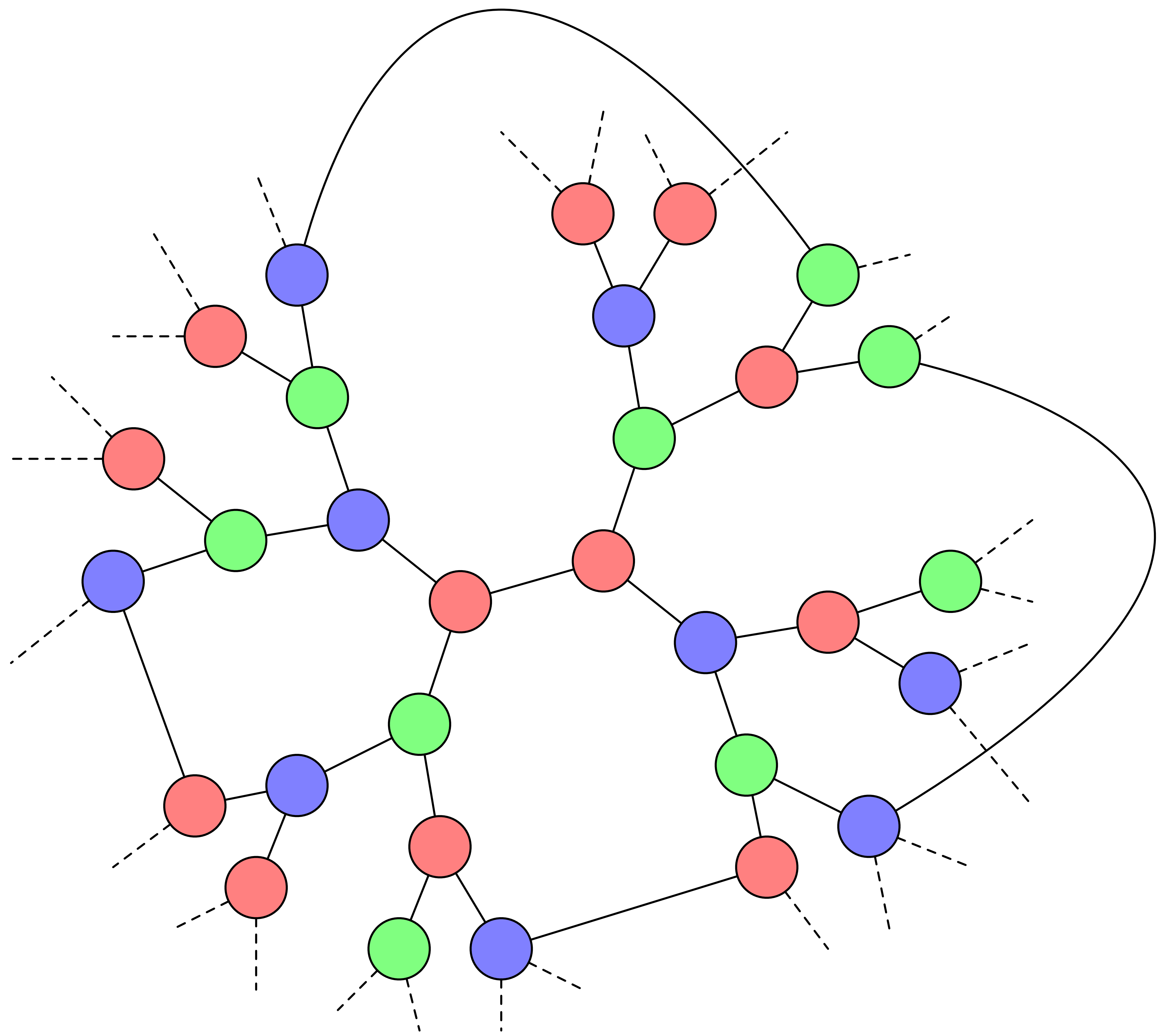

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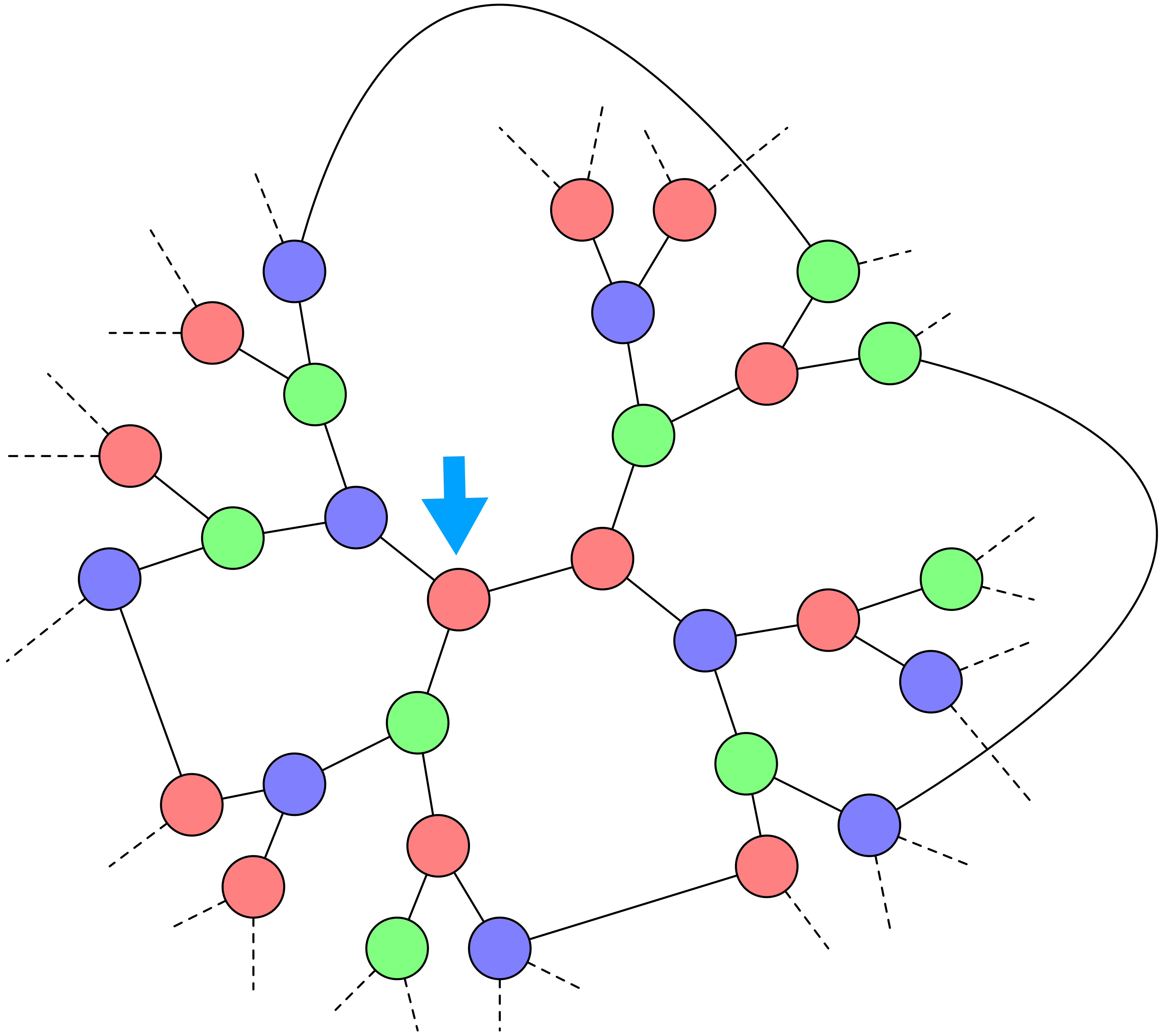

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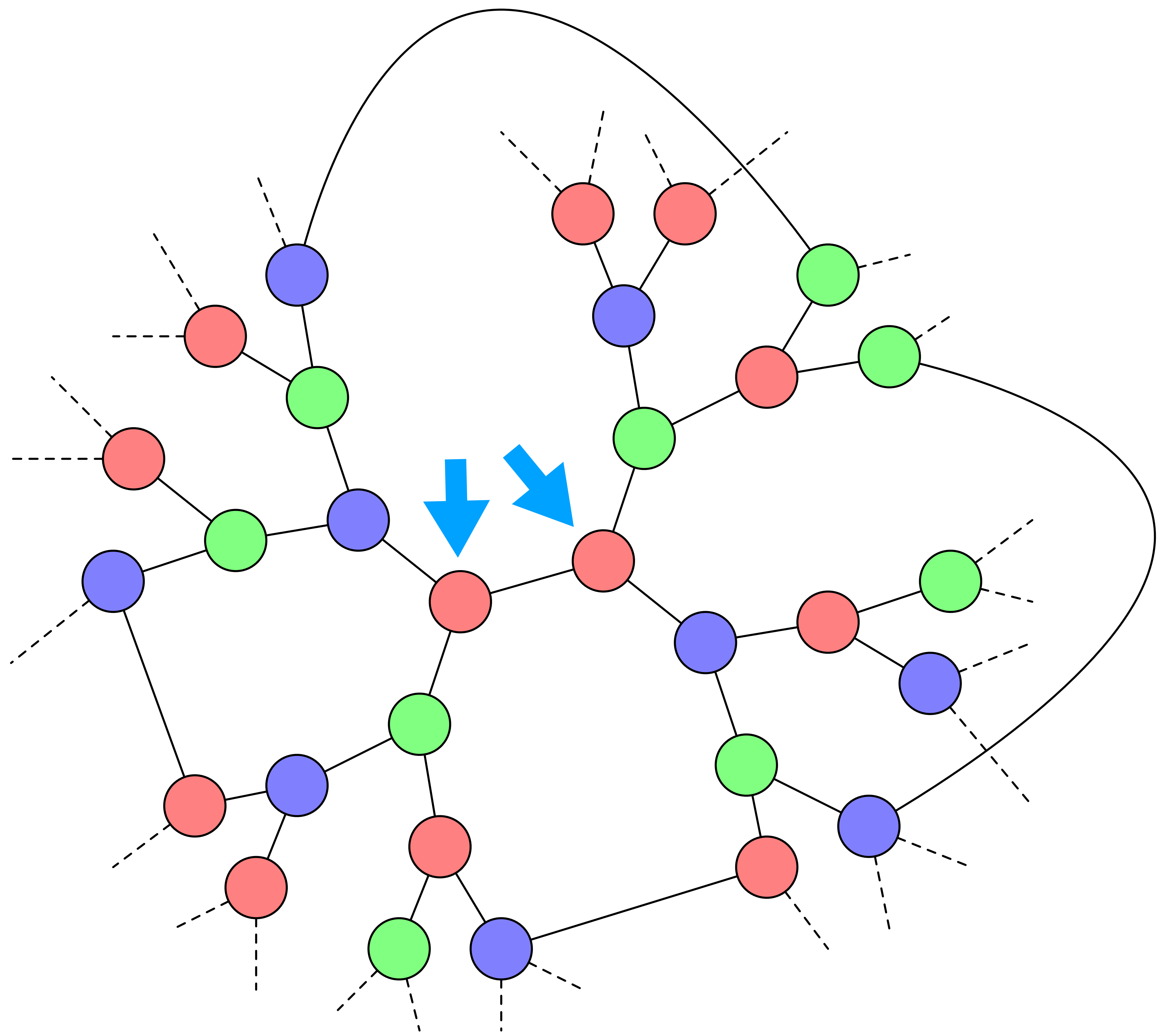

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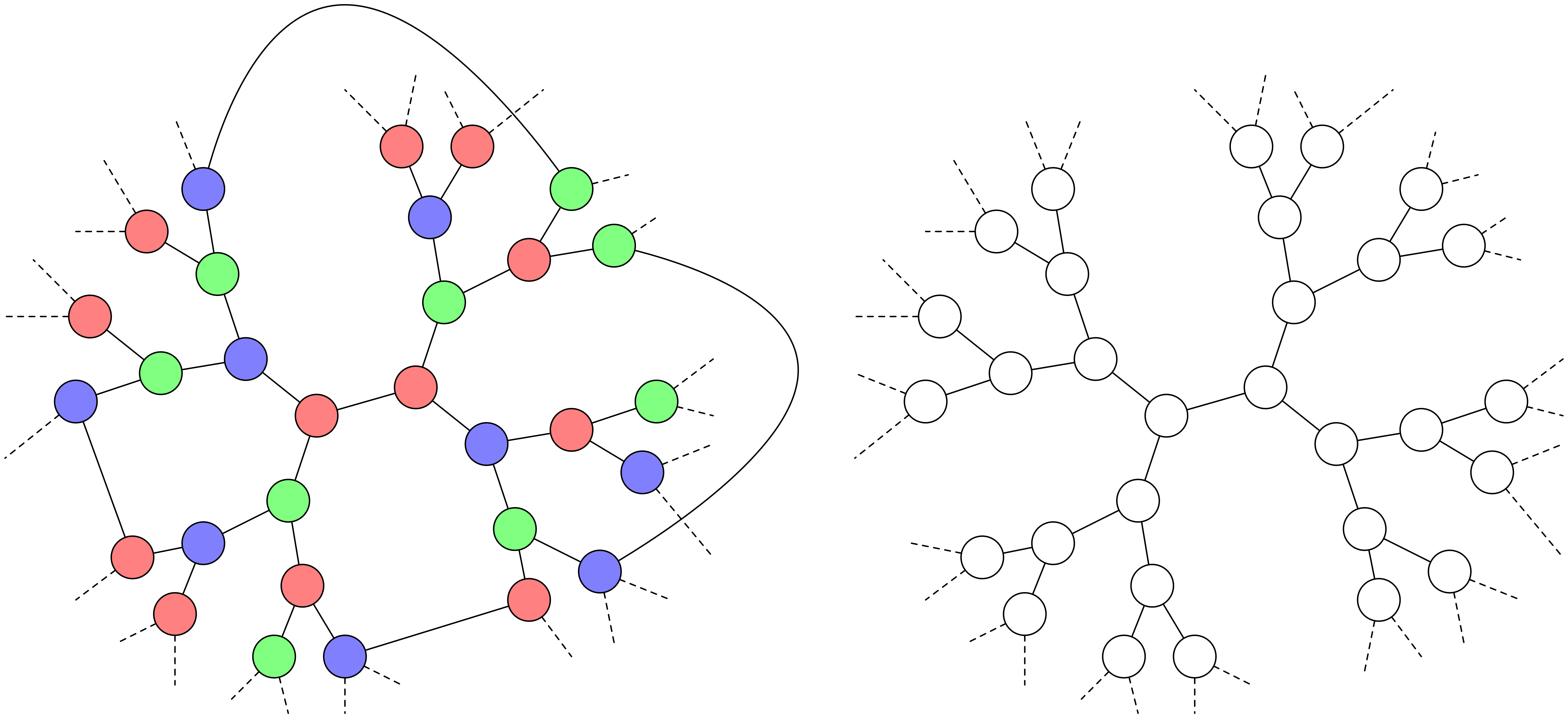

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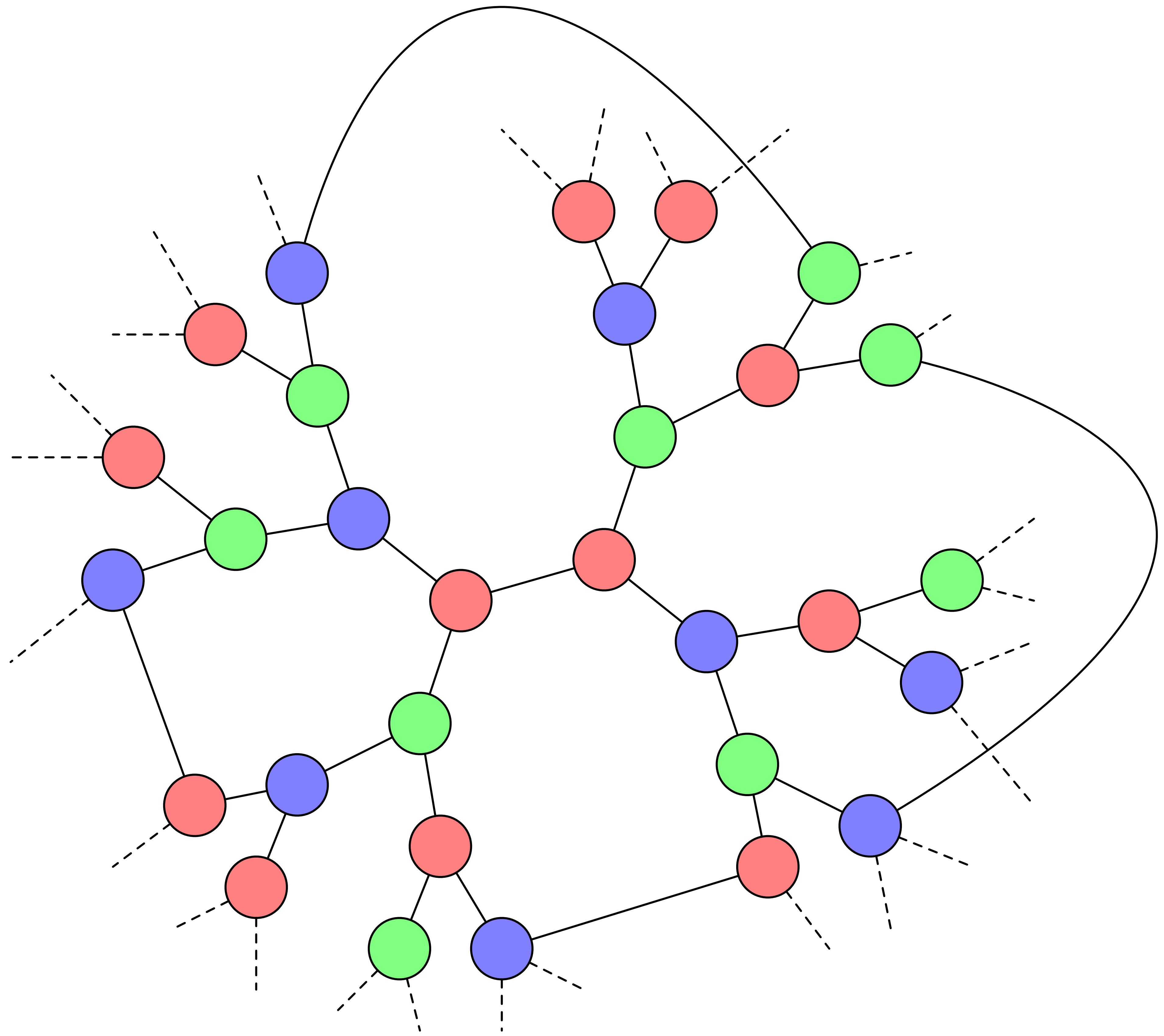
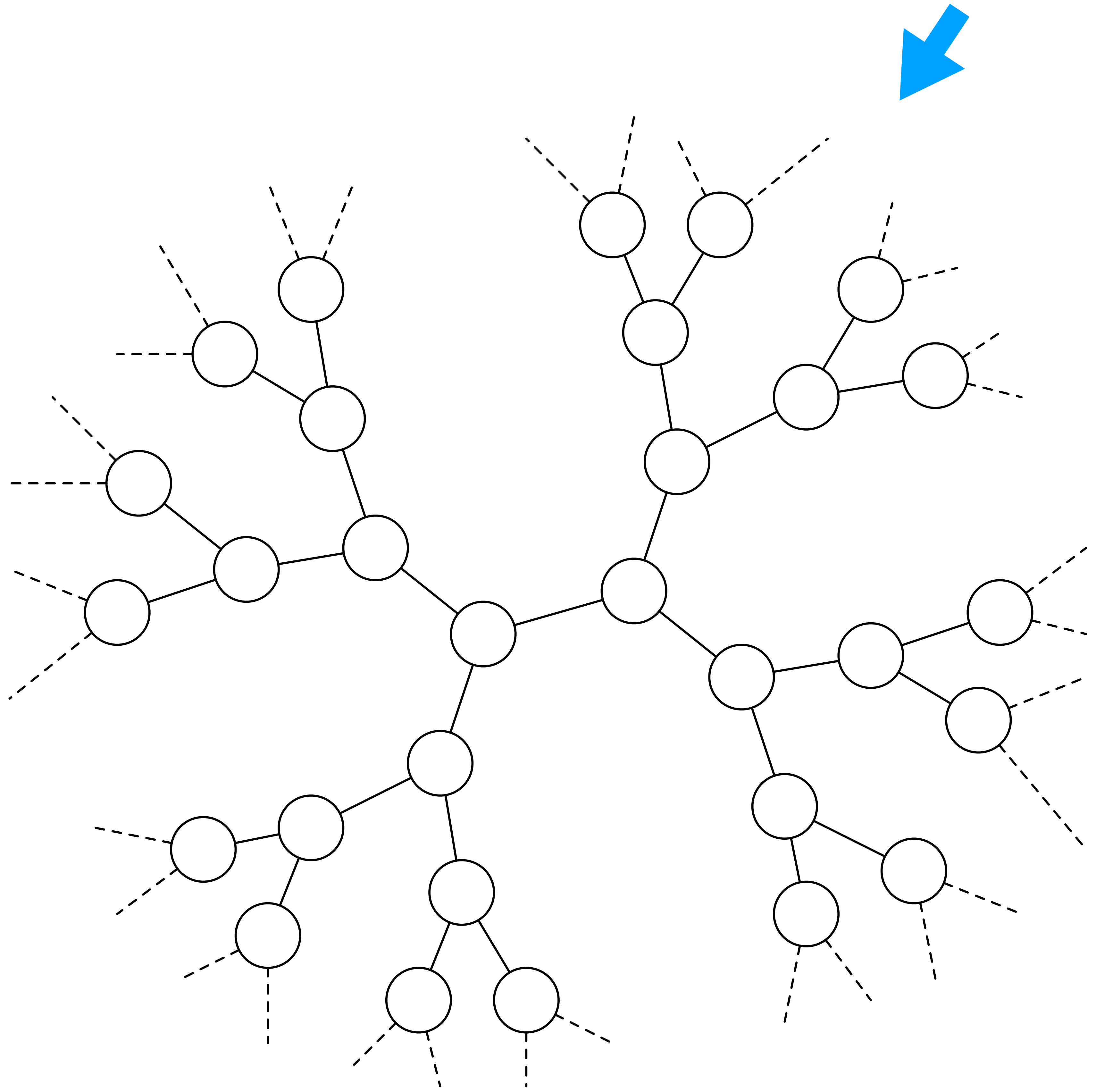


A real tree



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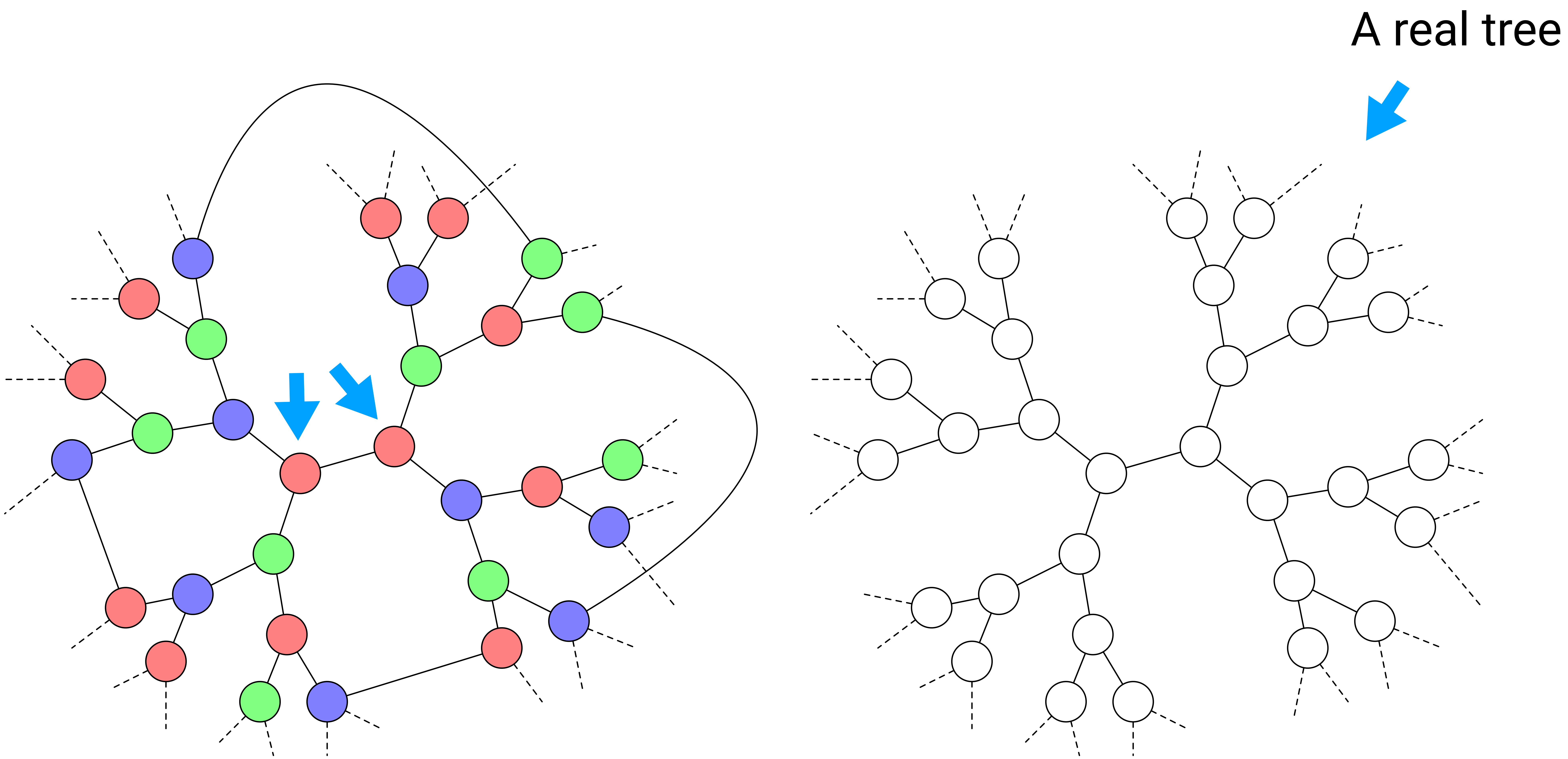

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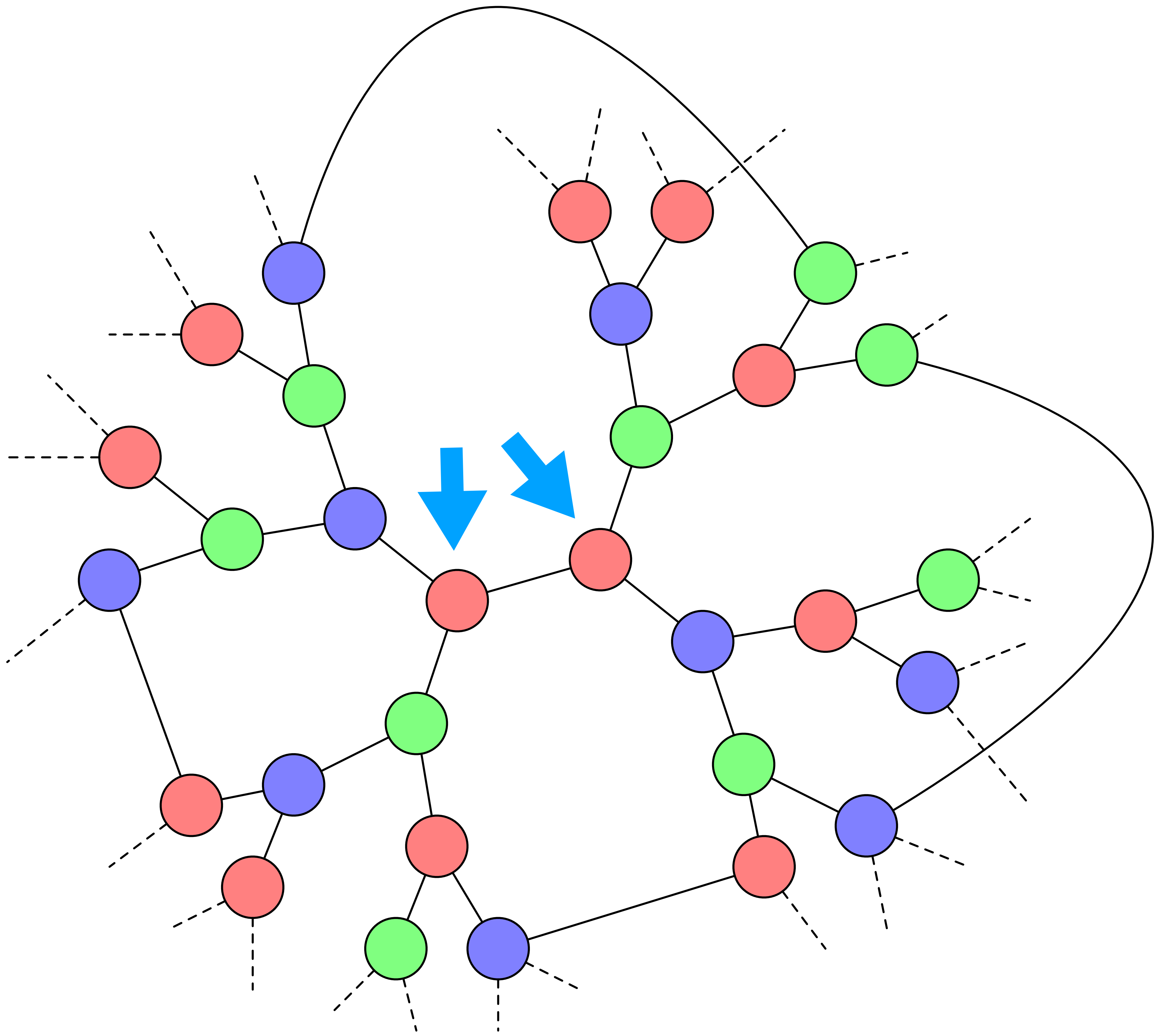

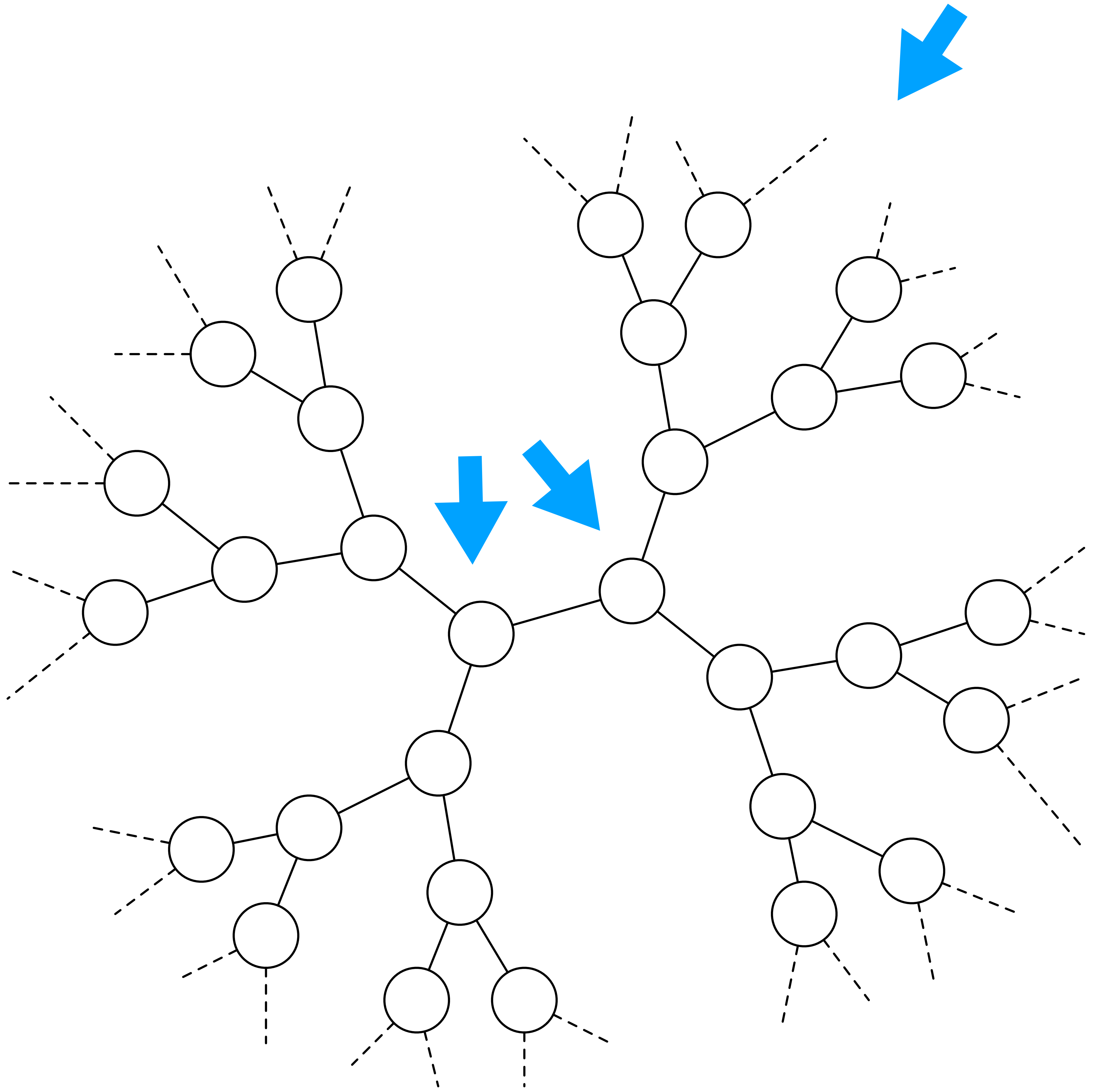

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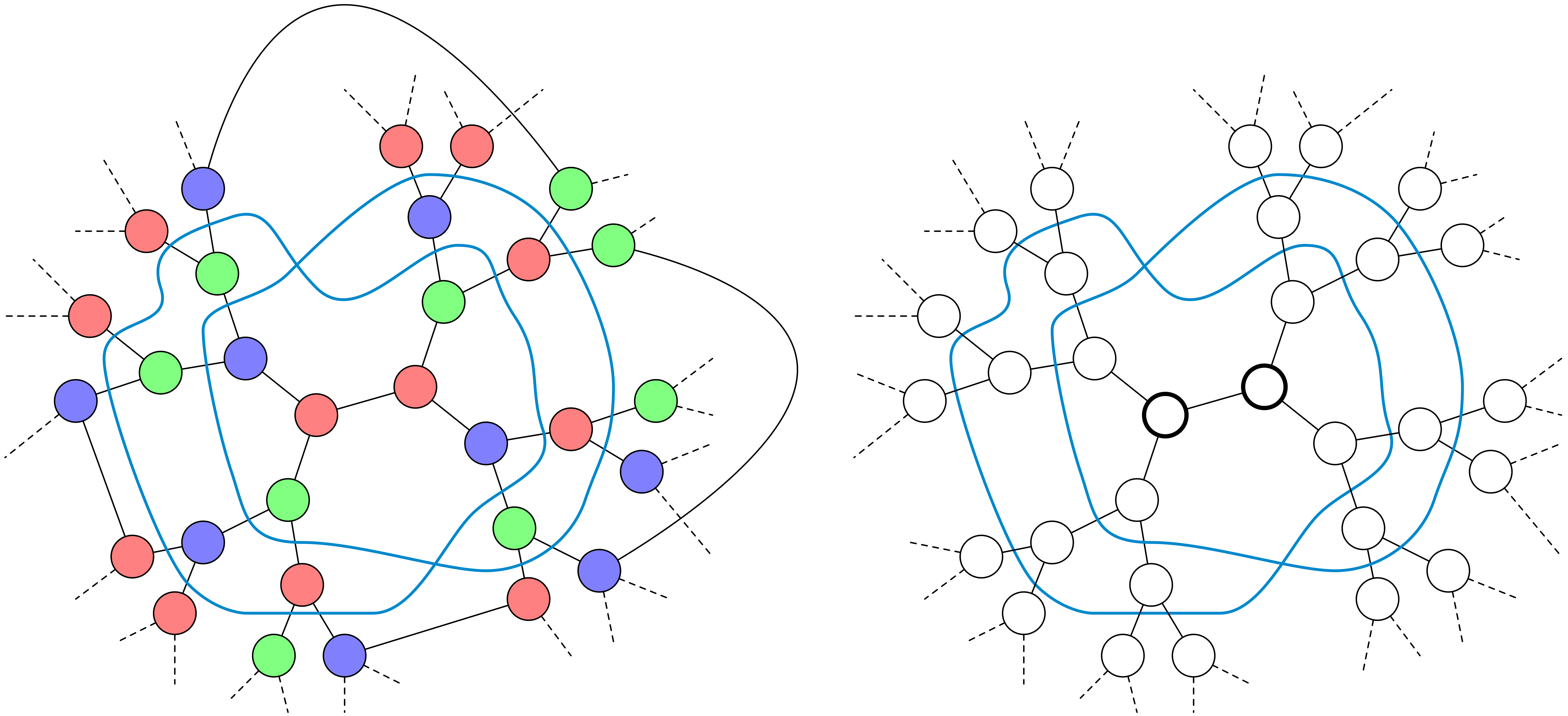

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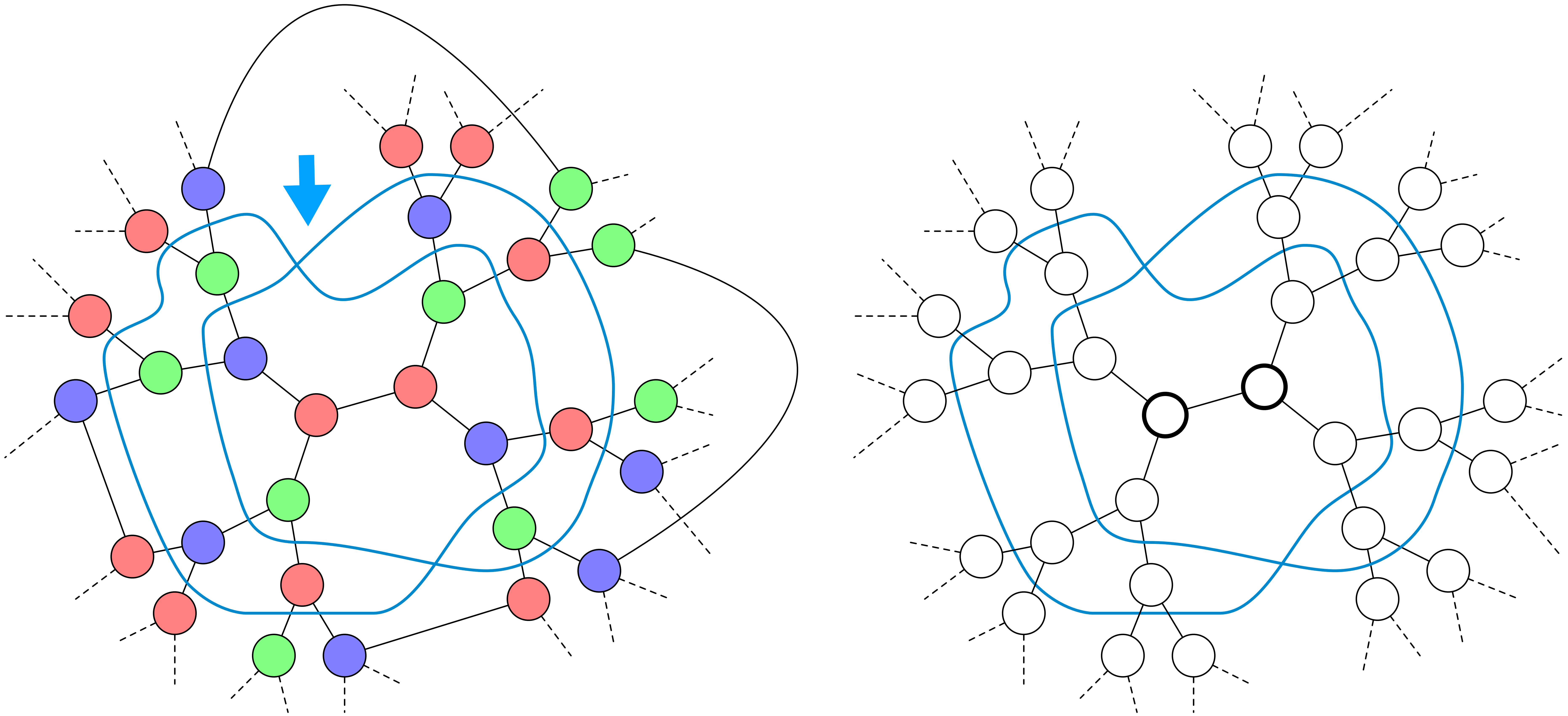

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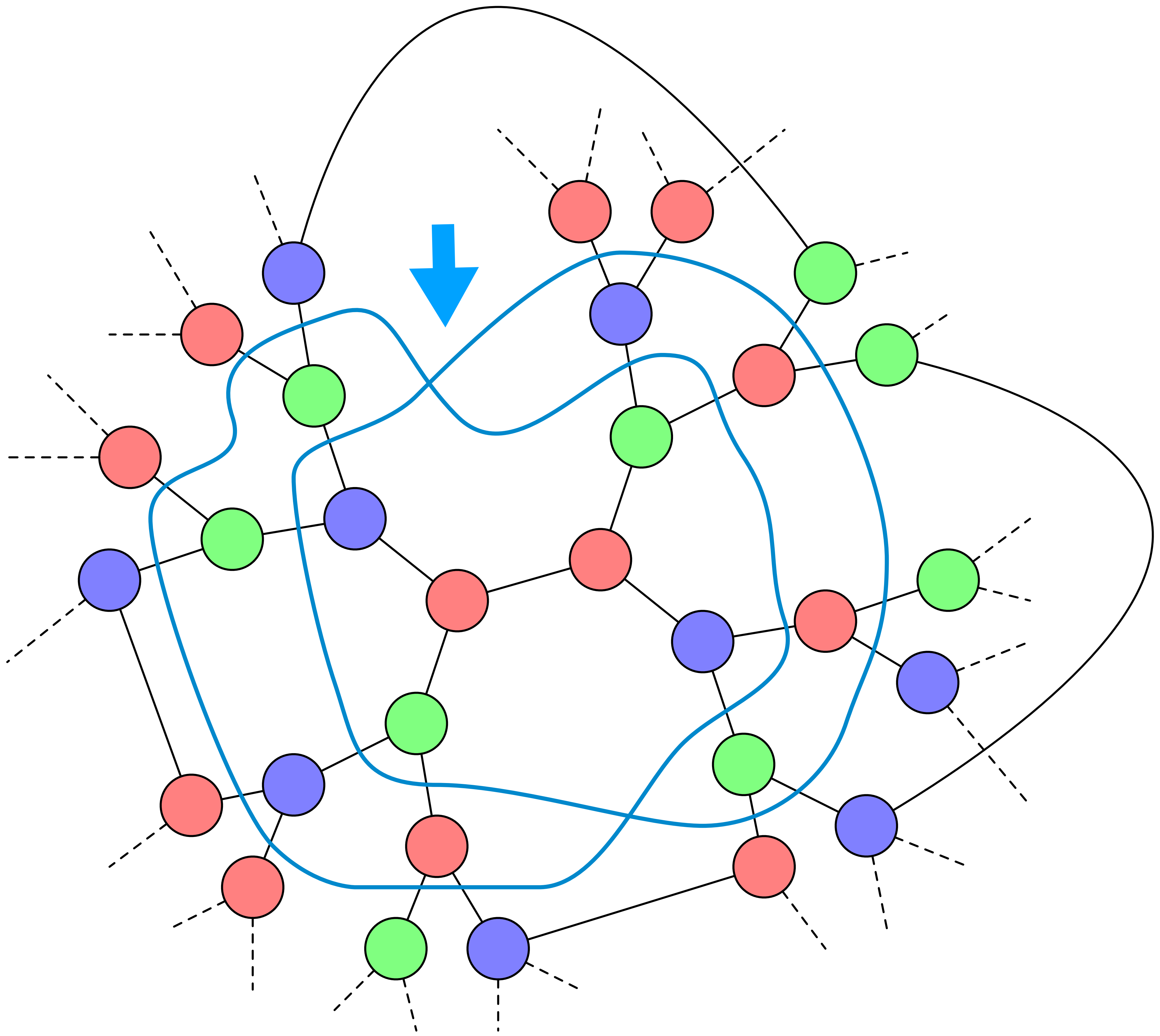

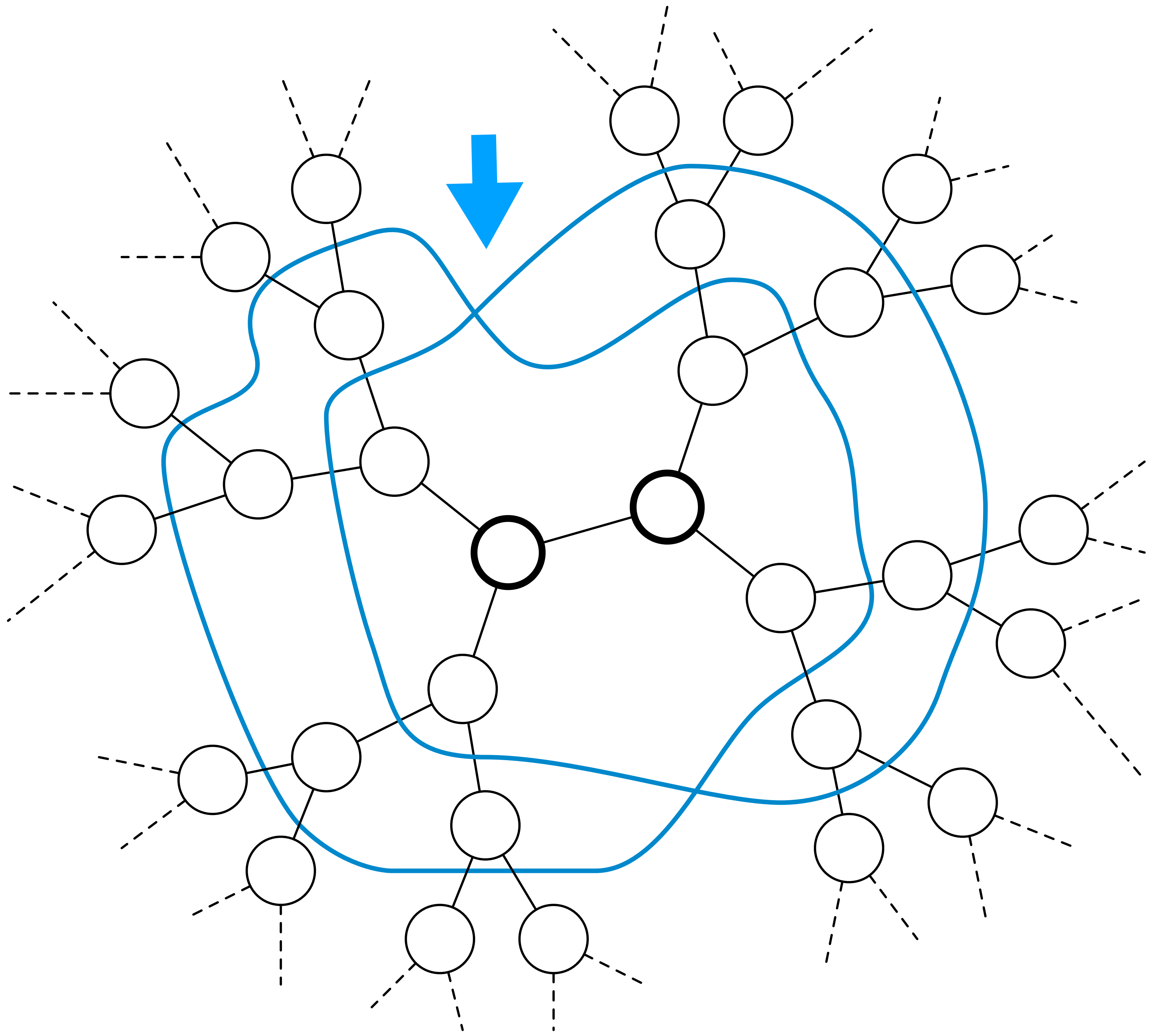

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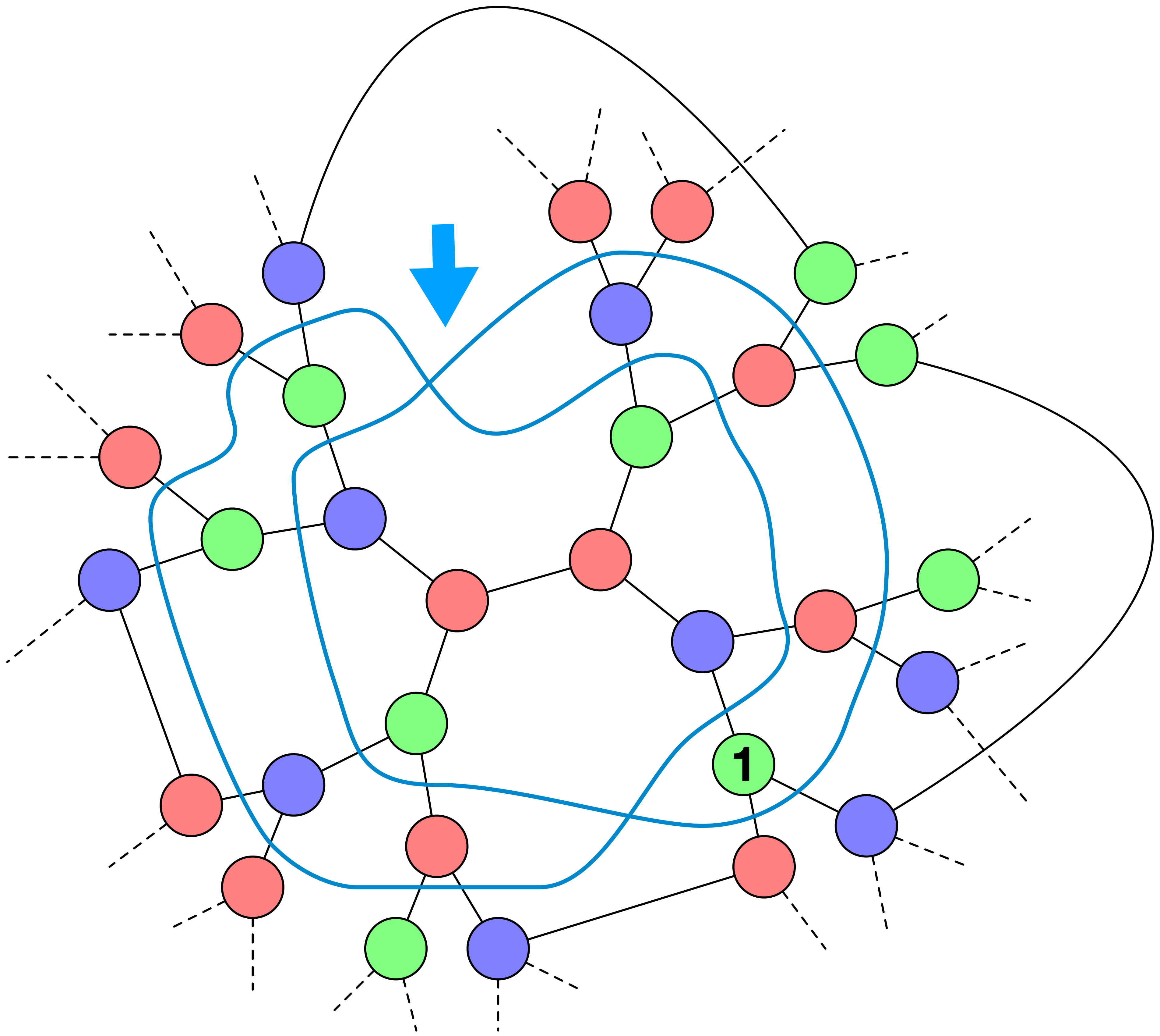

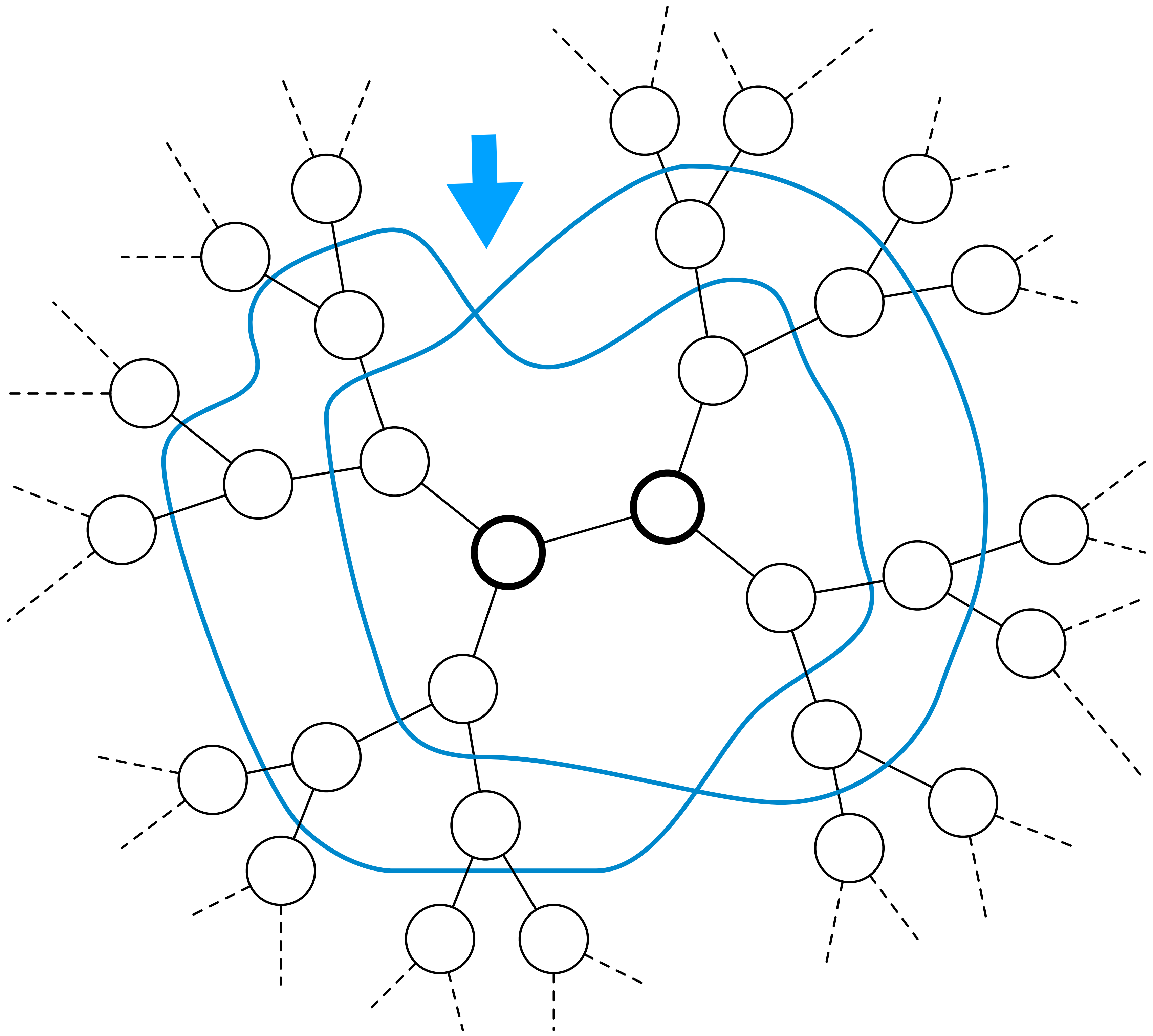

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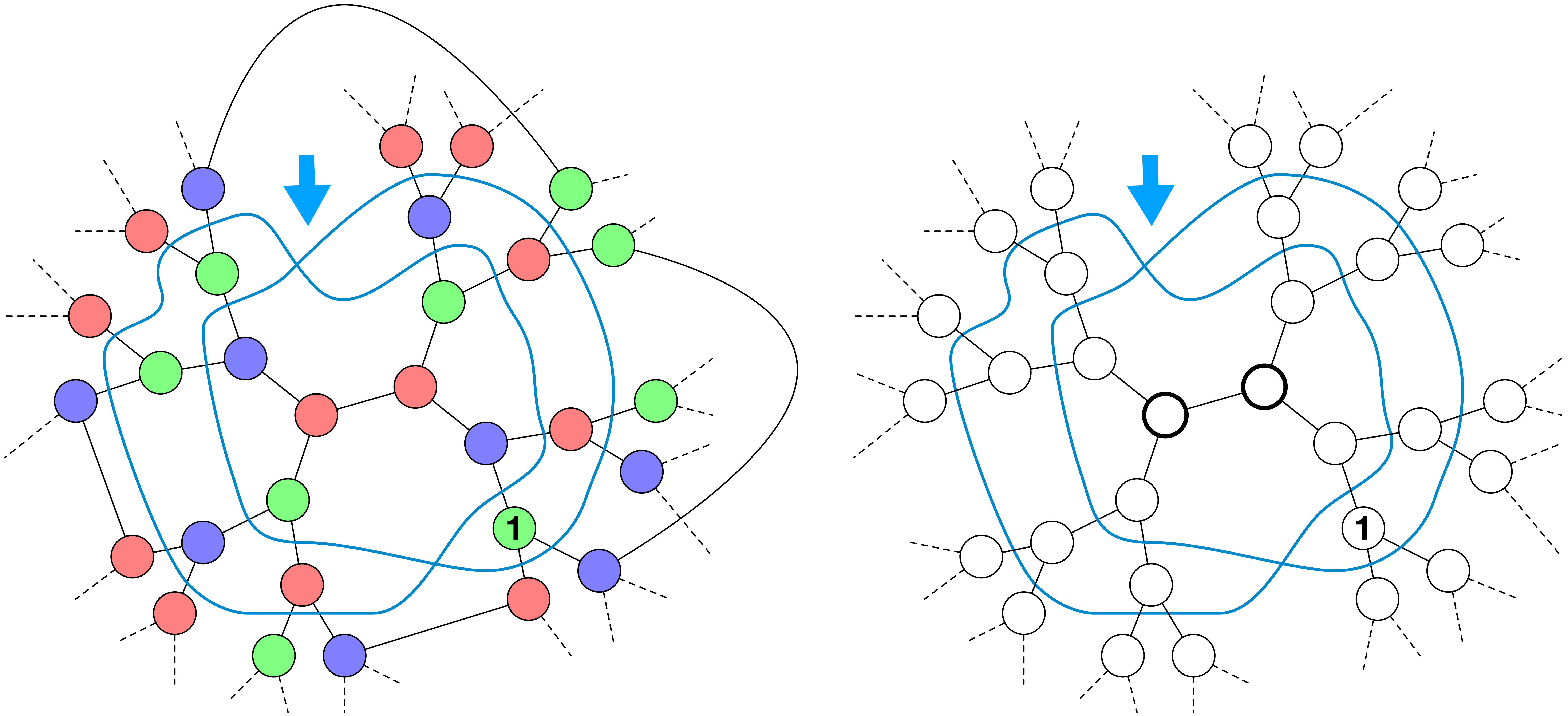

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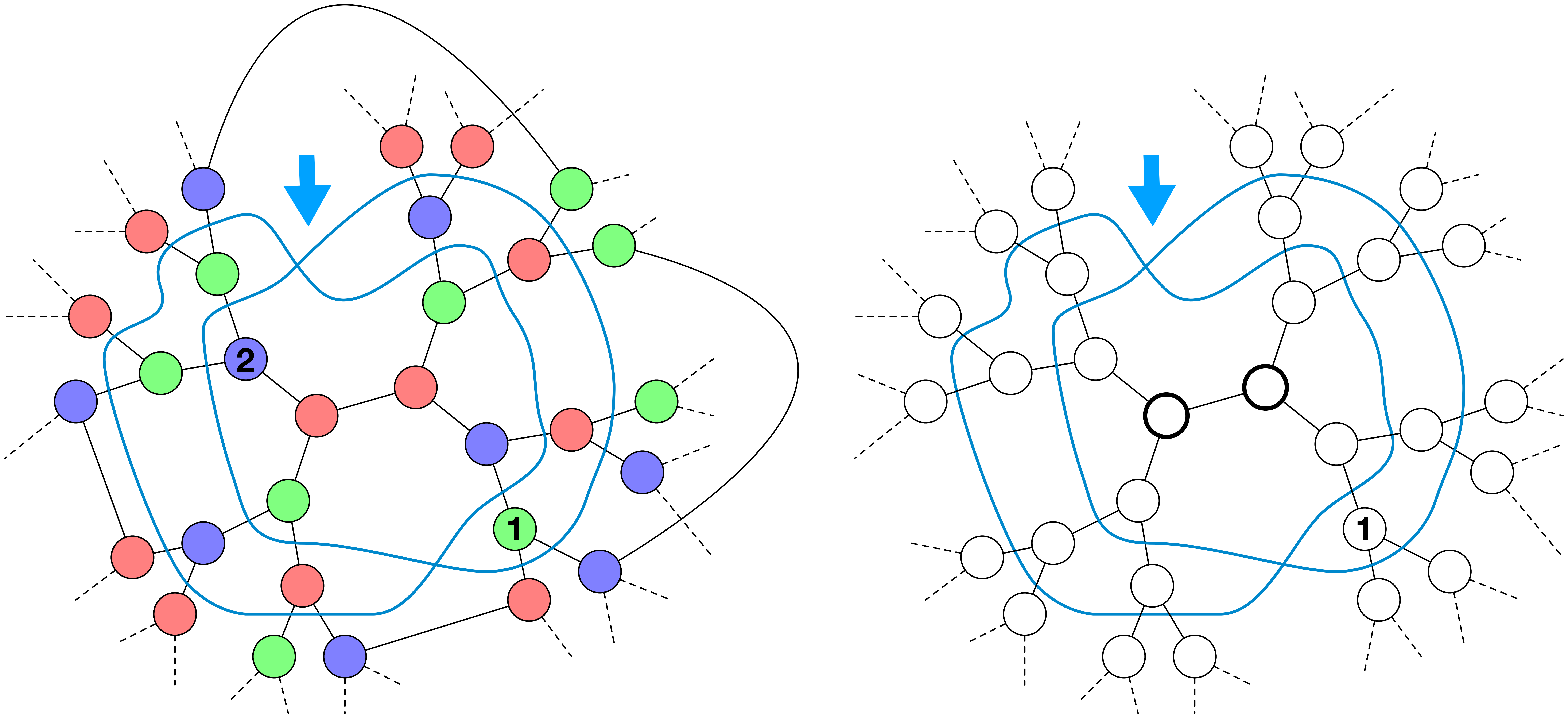

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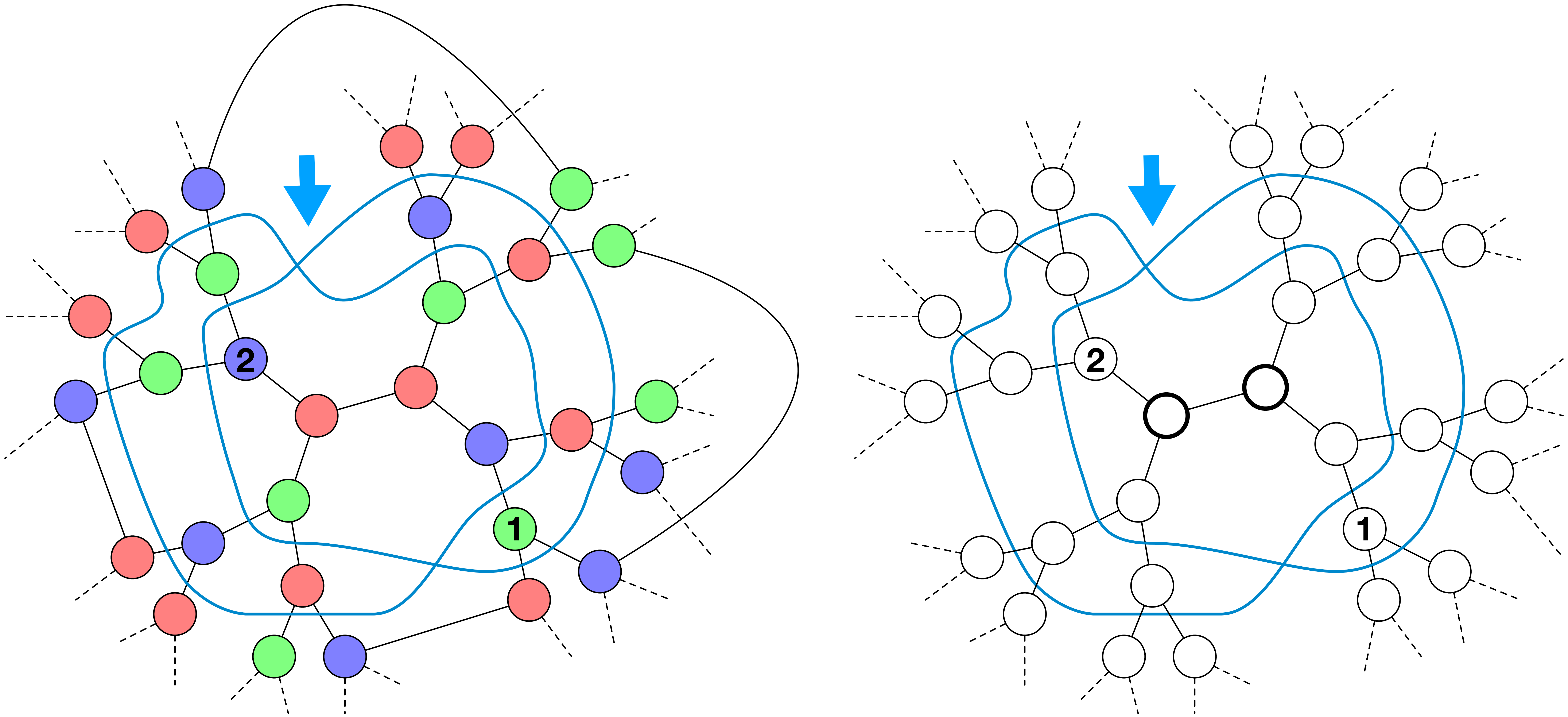

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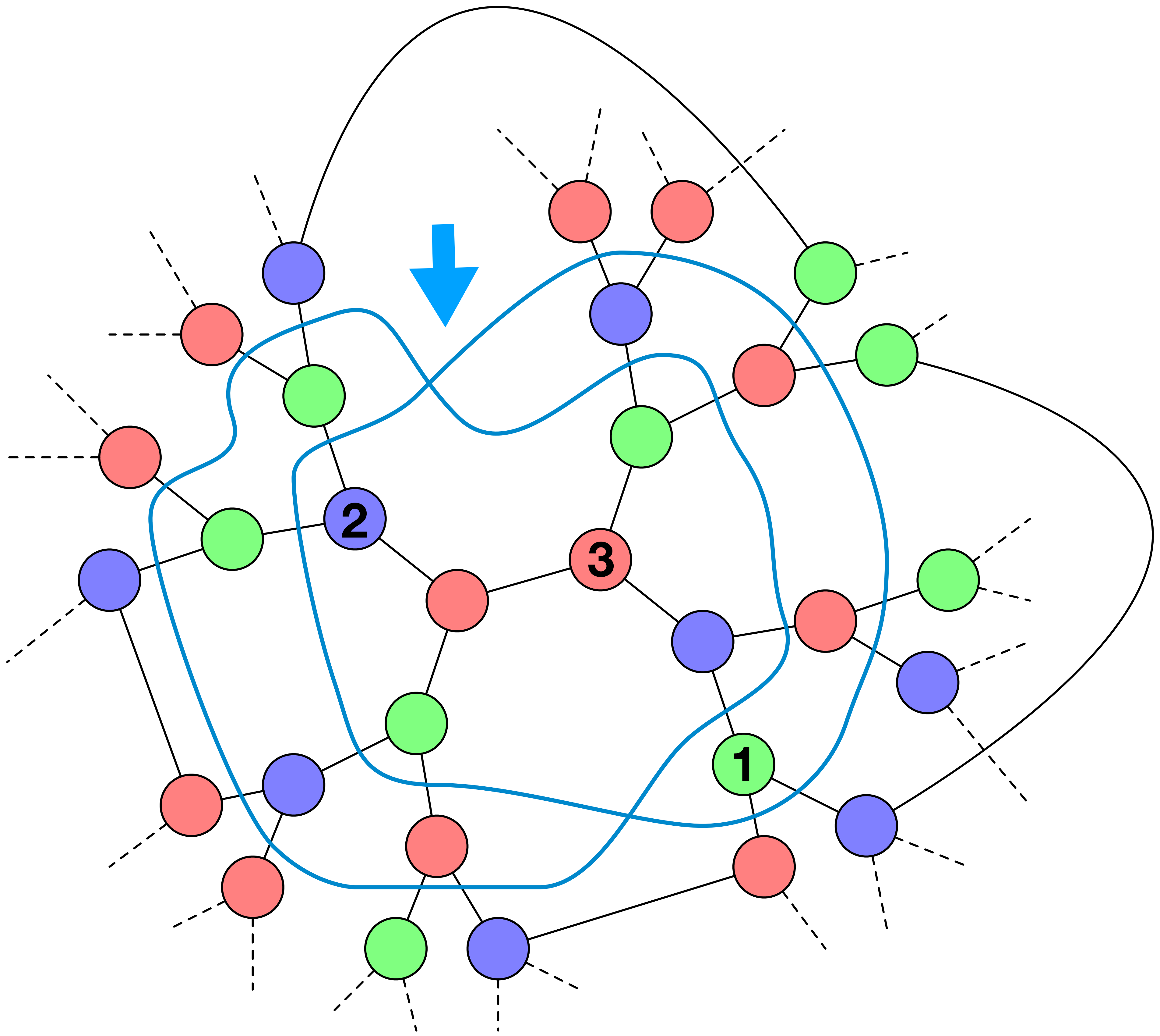

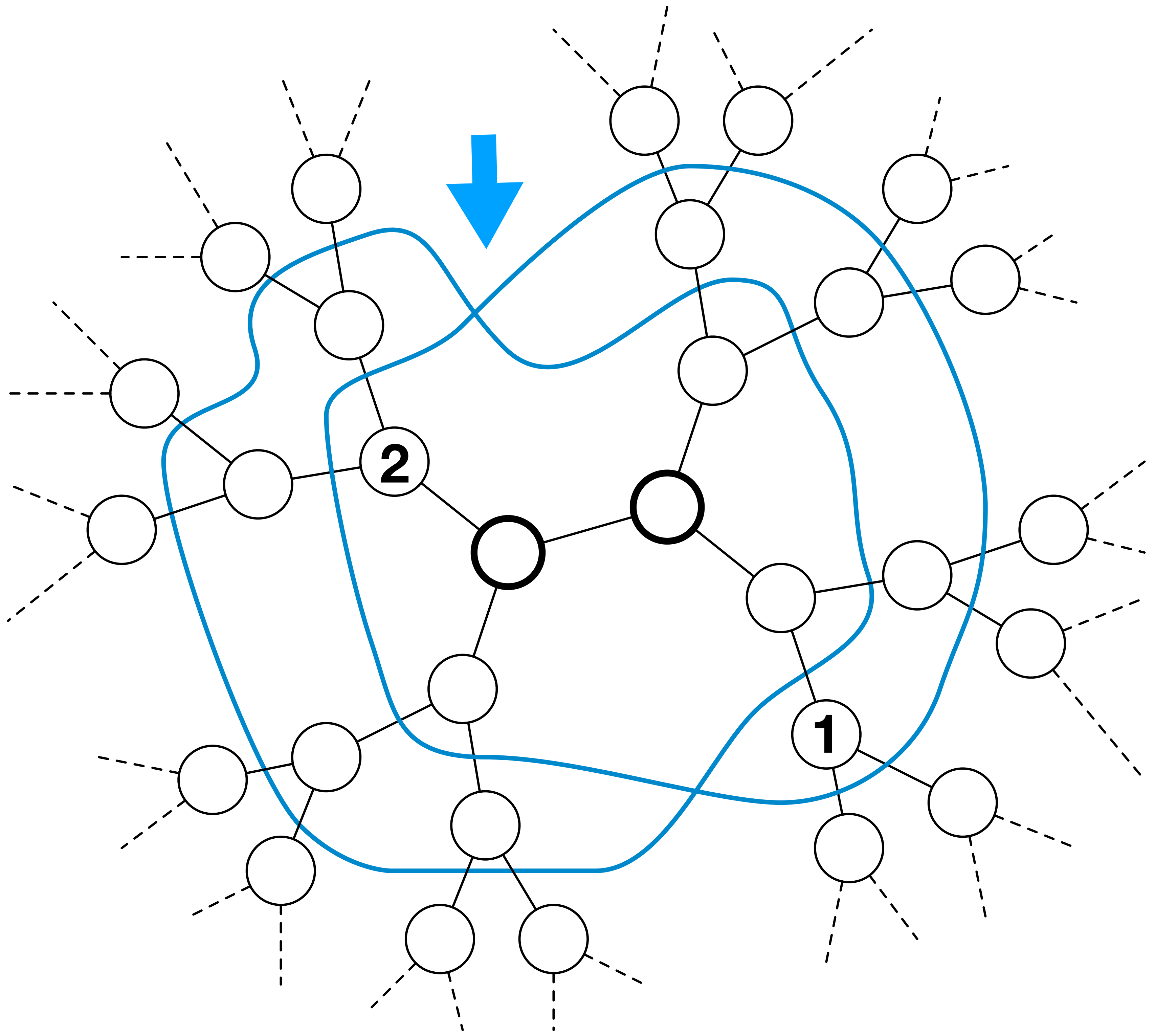

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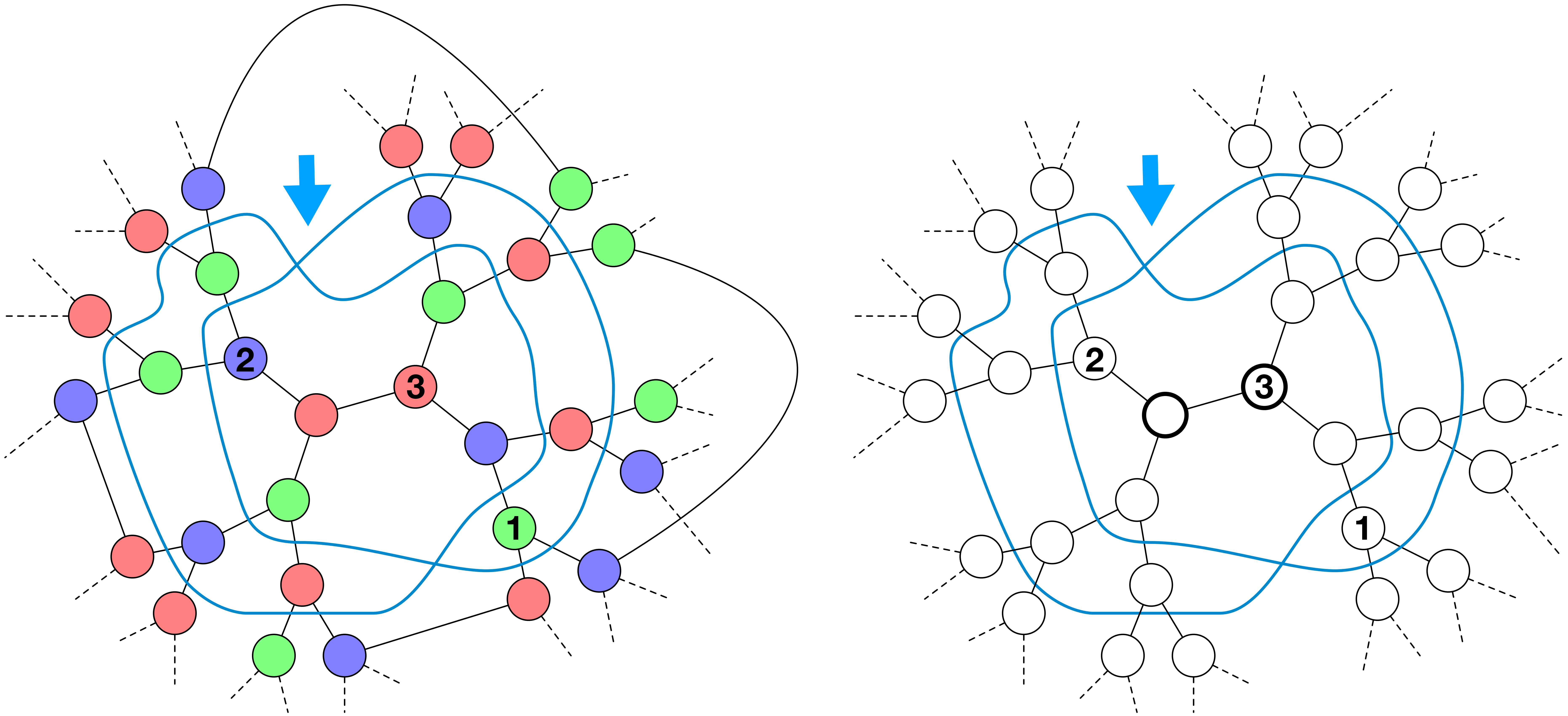

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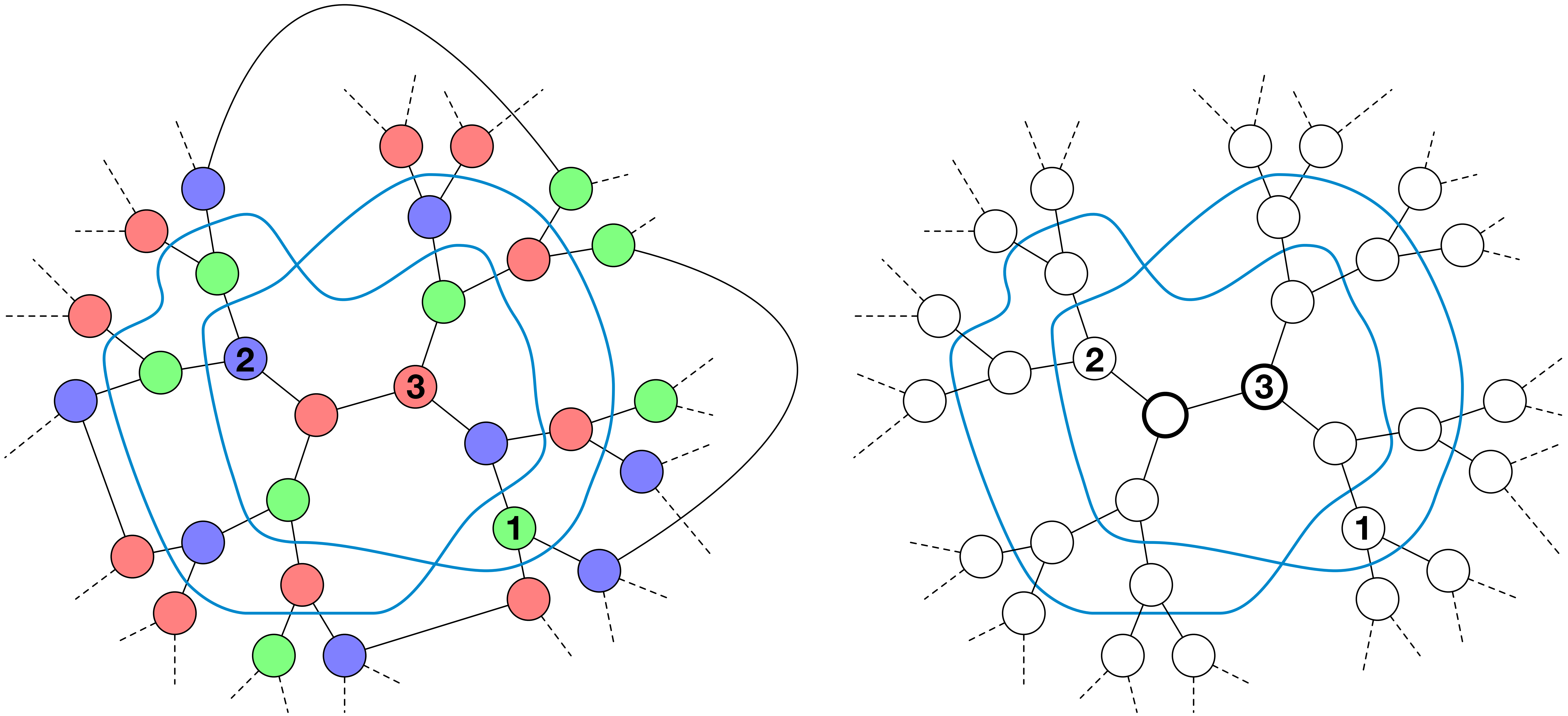

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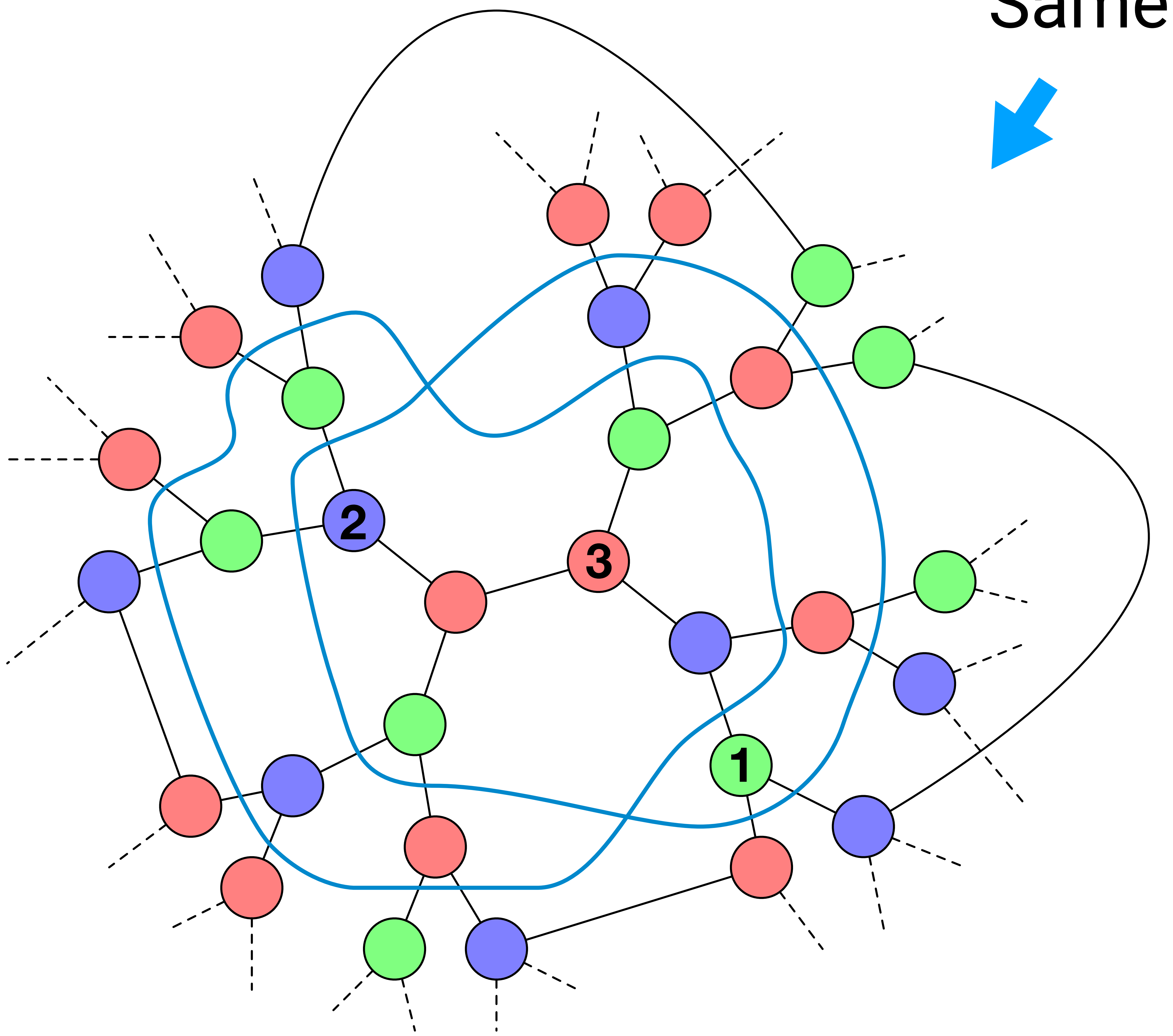
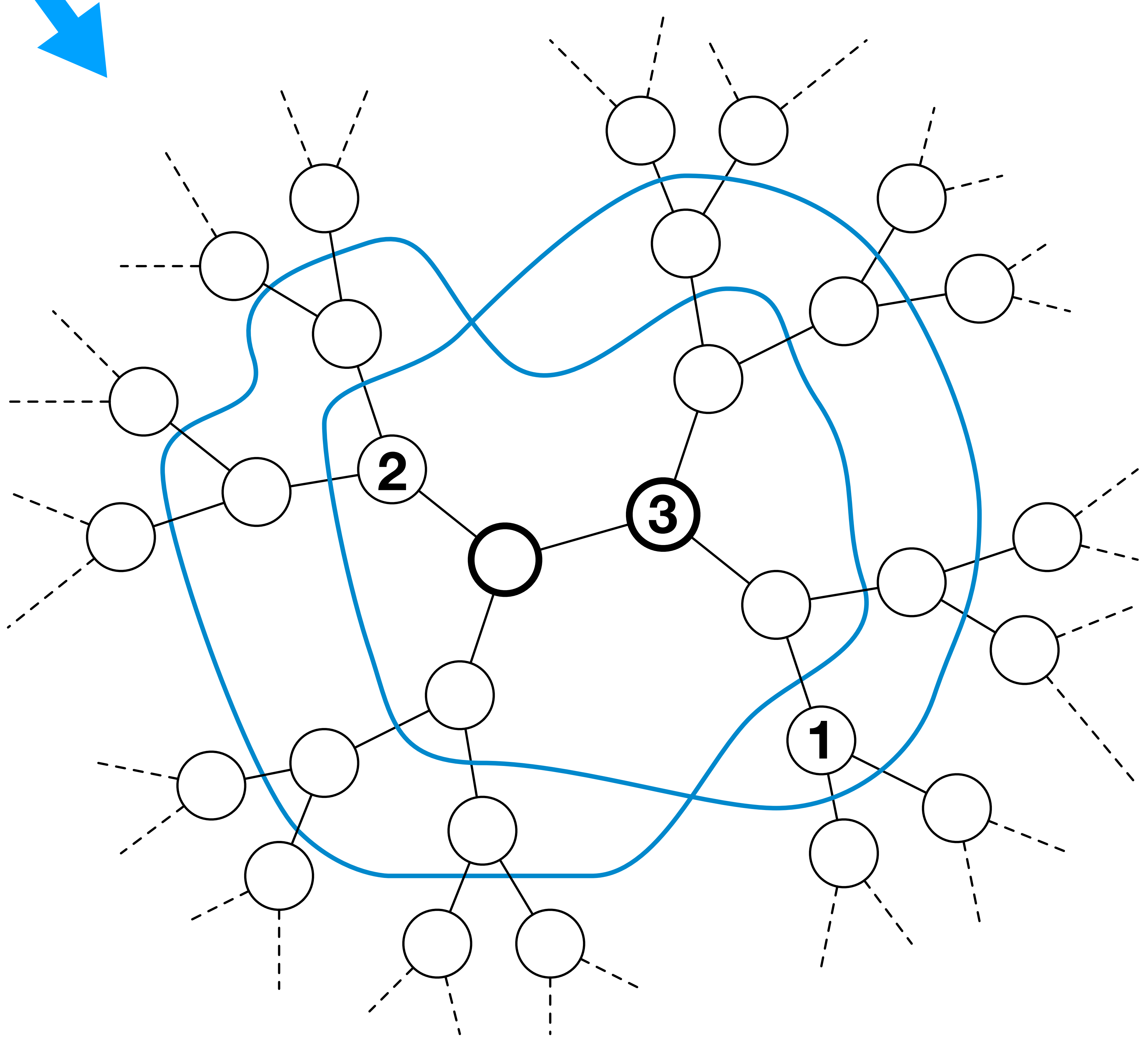
Same size





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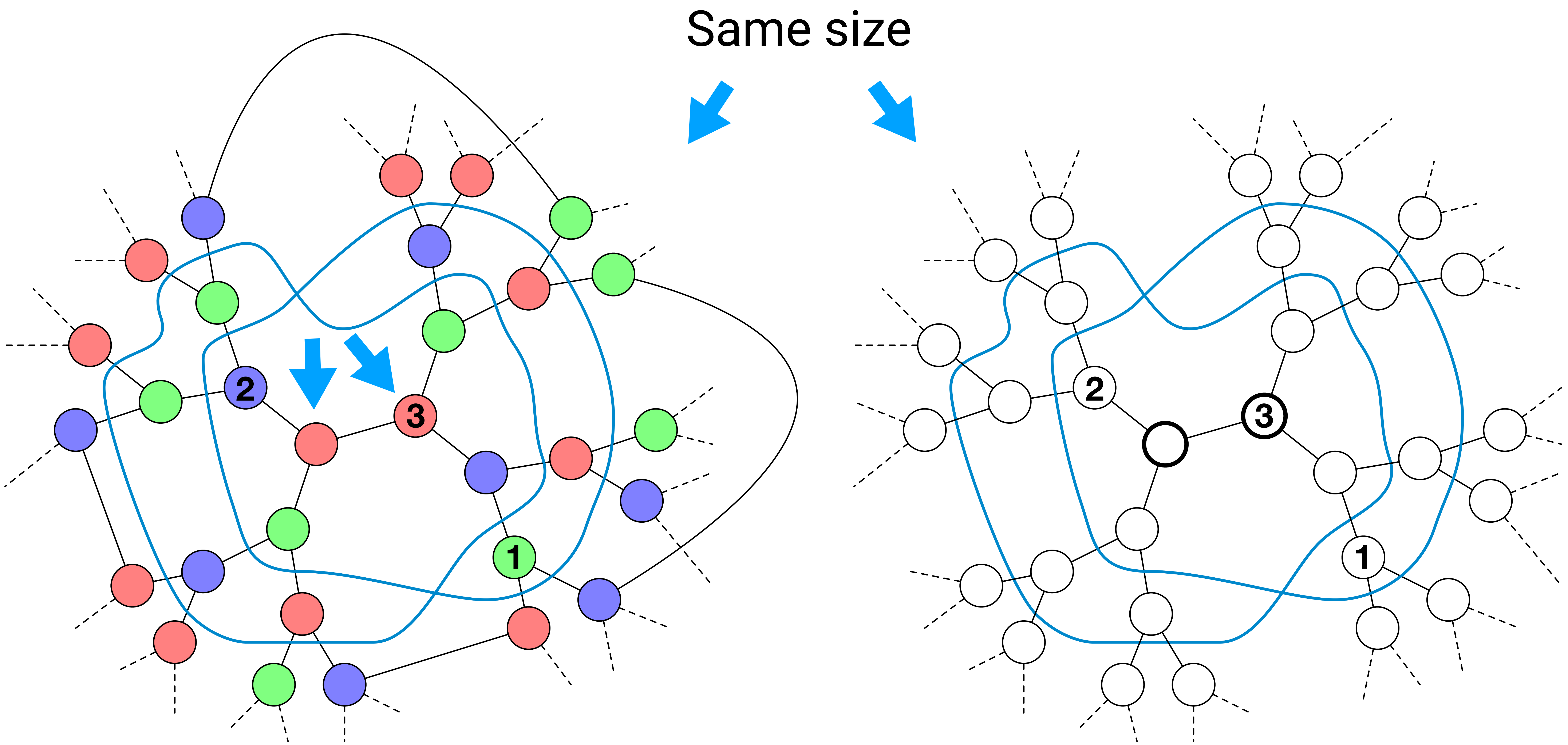

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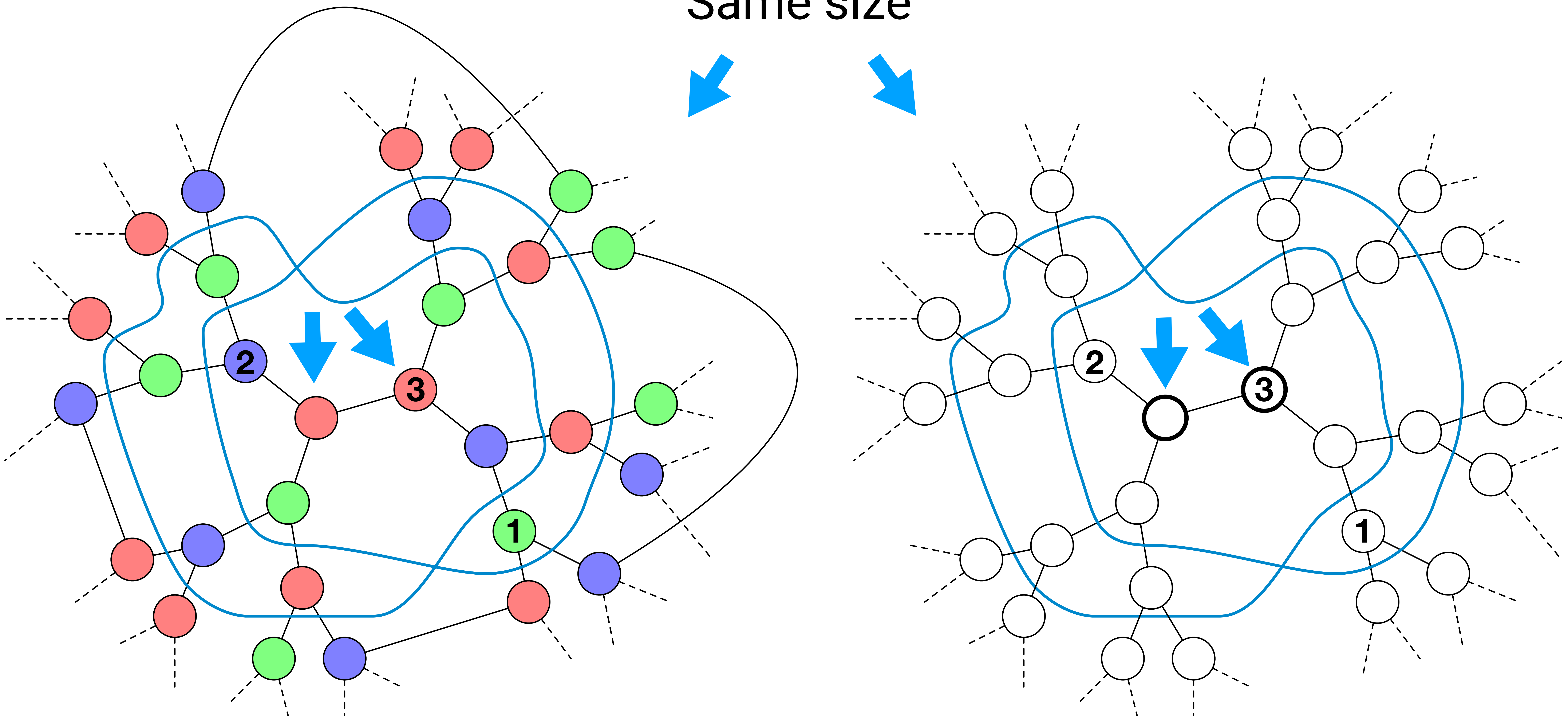

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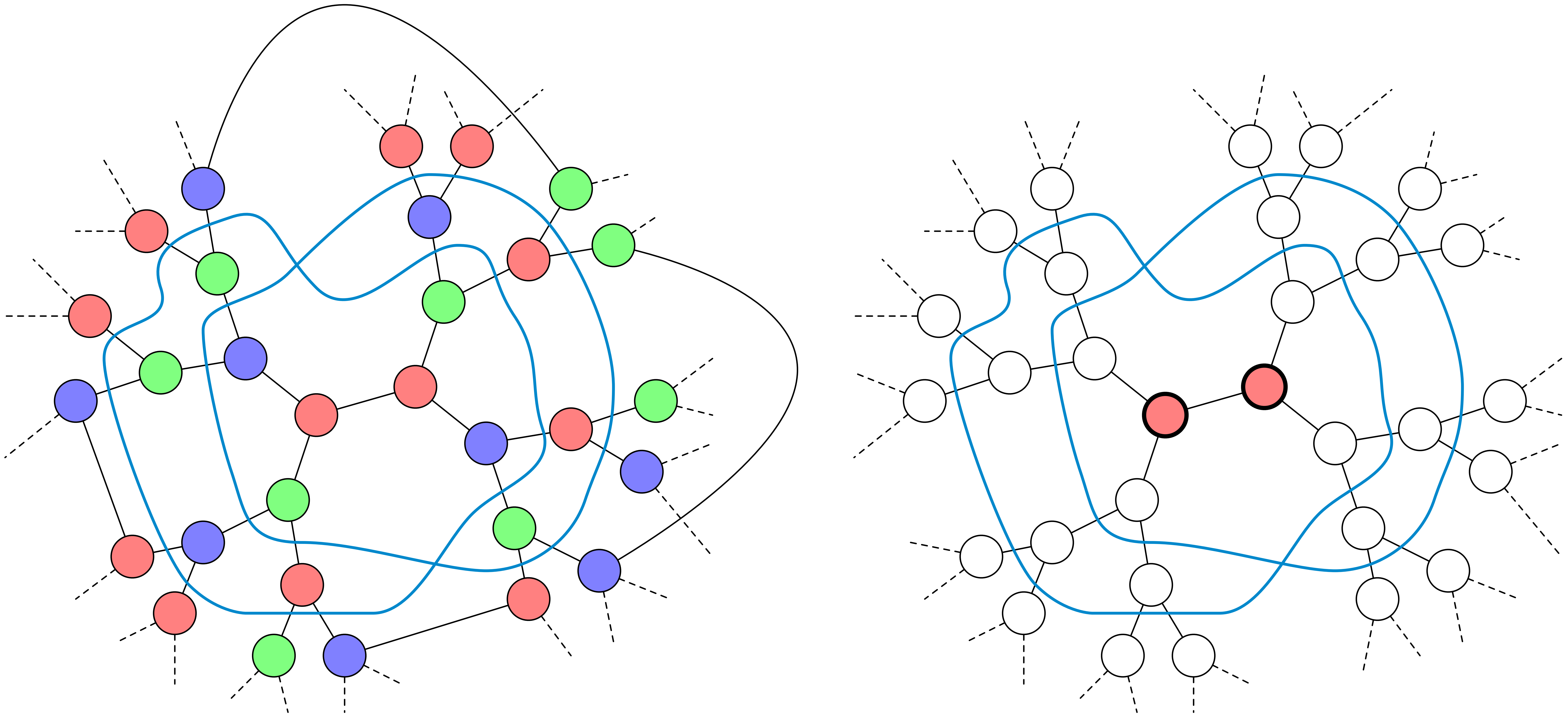

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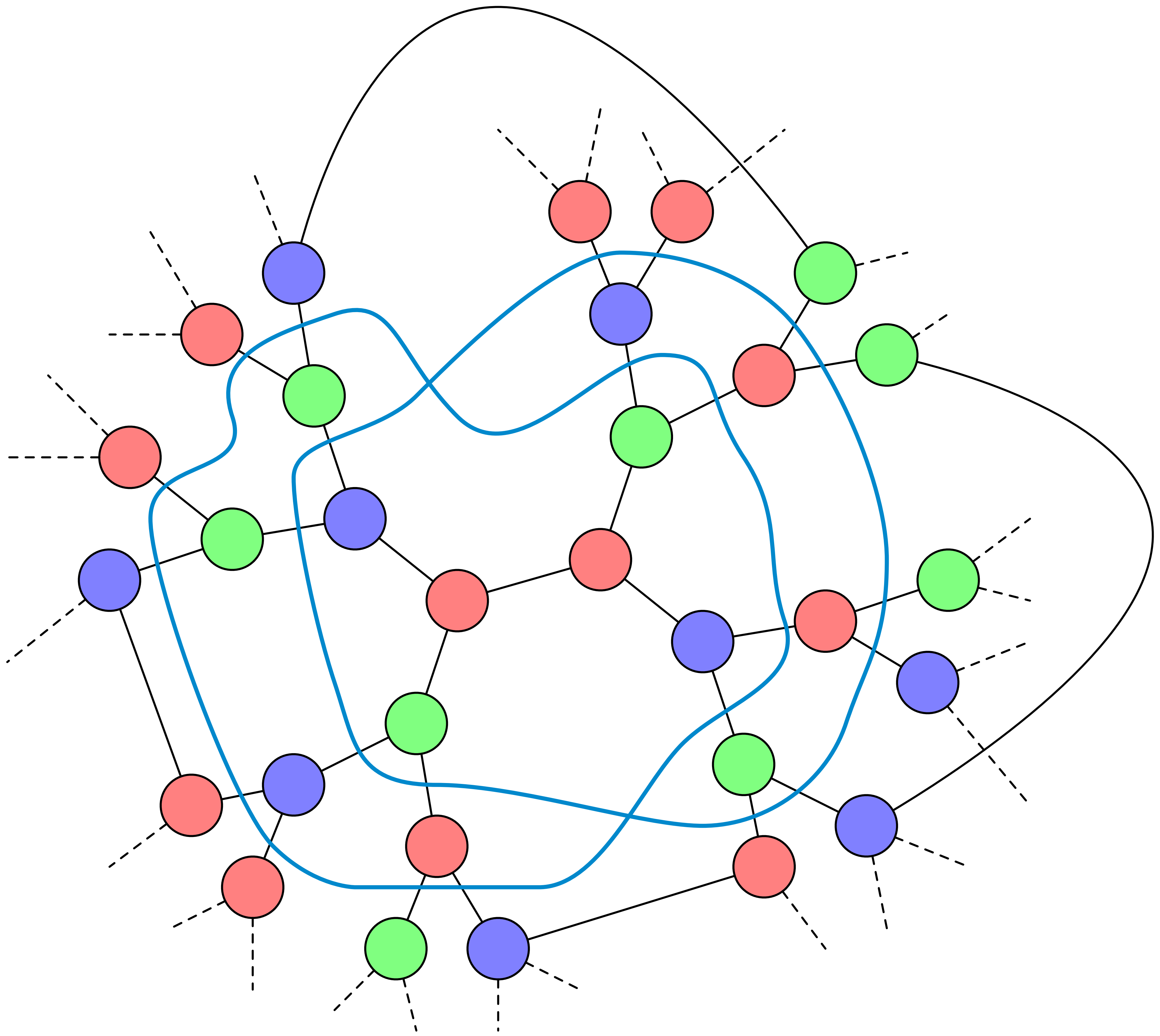
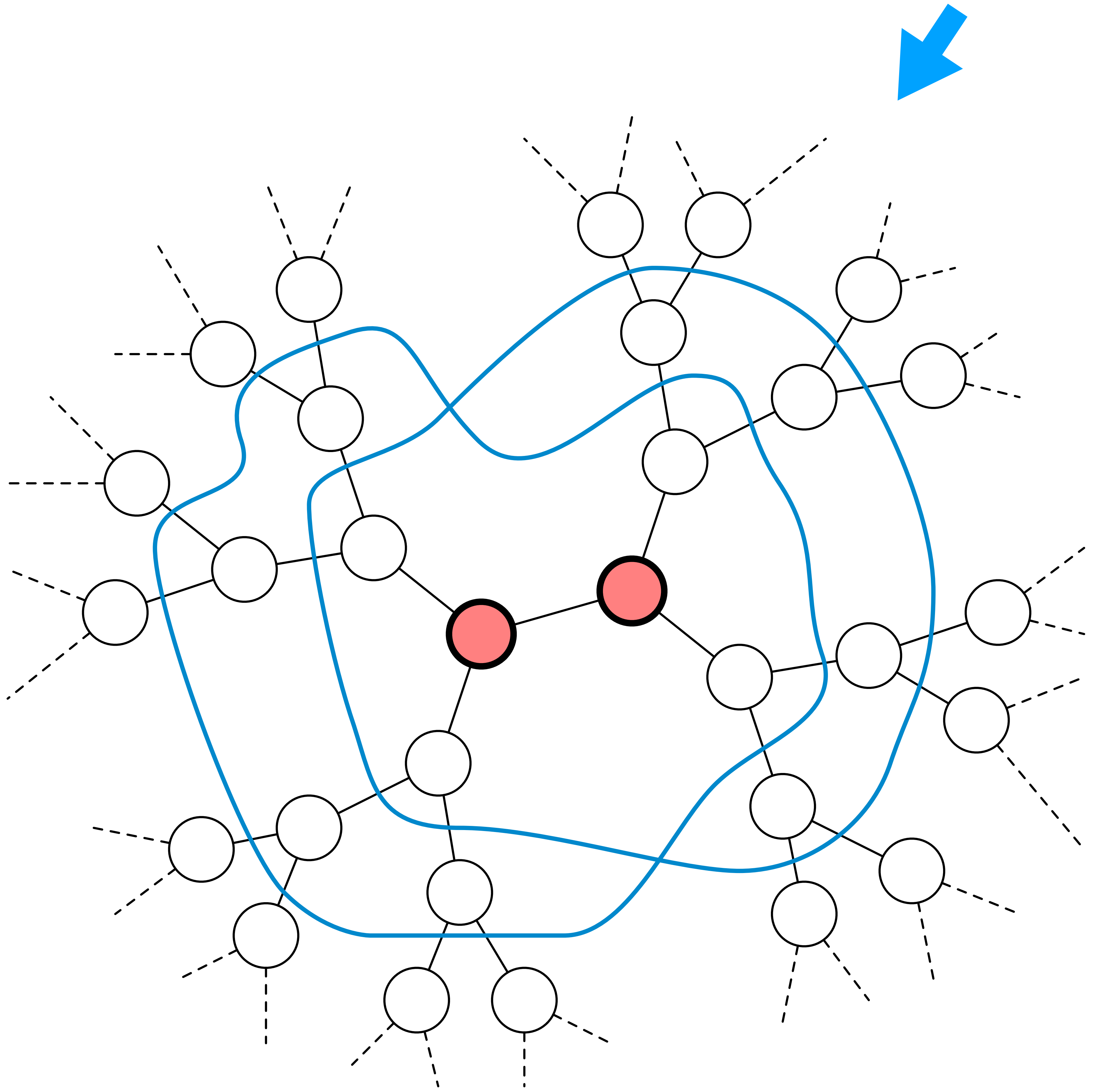
Fail on a real tree!





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  - Note that, if **$\Delta=O(1)$**, then **$(\Delta+1)$-coloring** can be solved in just **$O(\log^* n)$**

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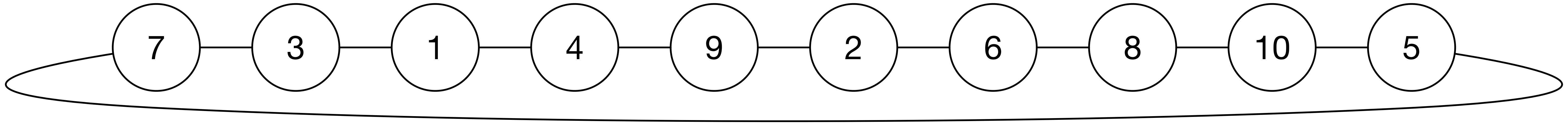

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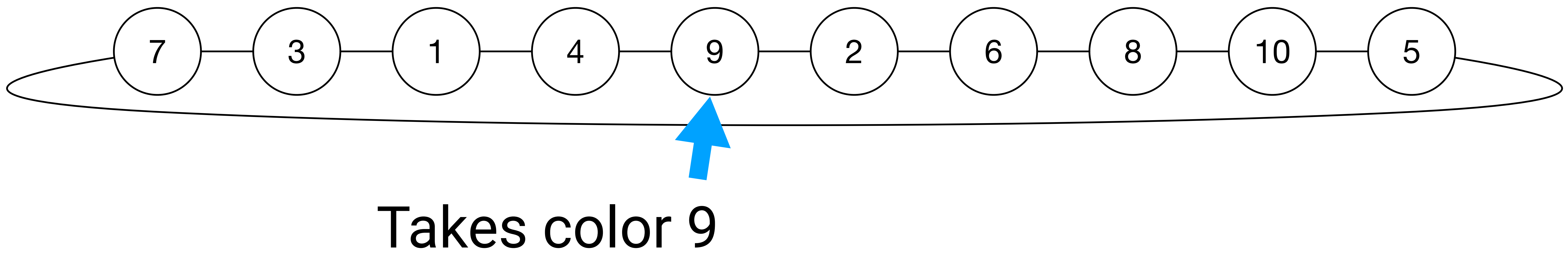

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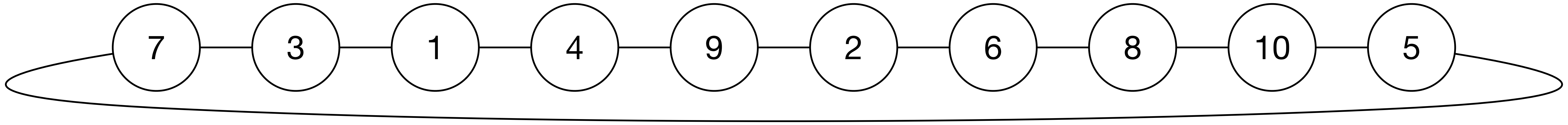

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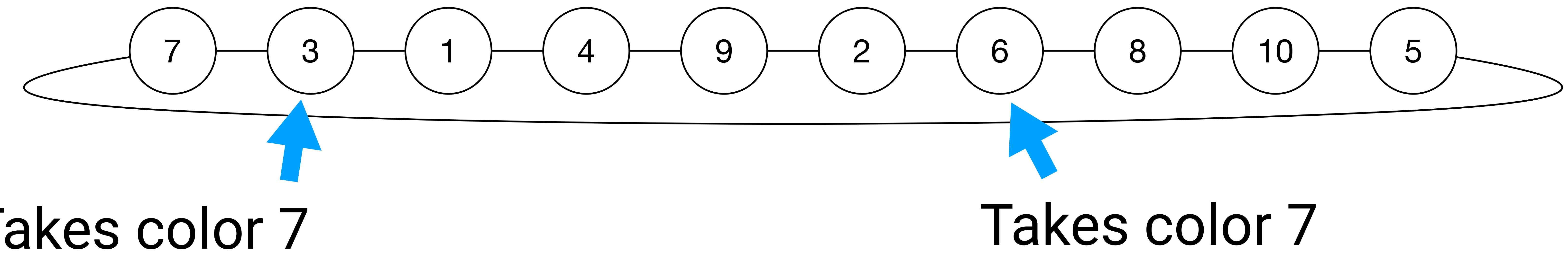

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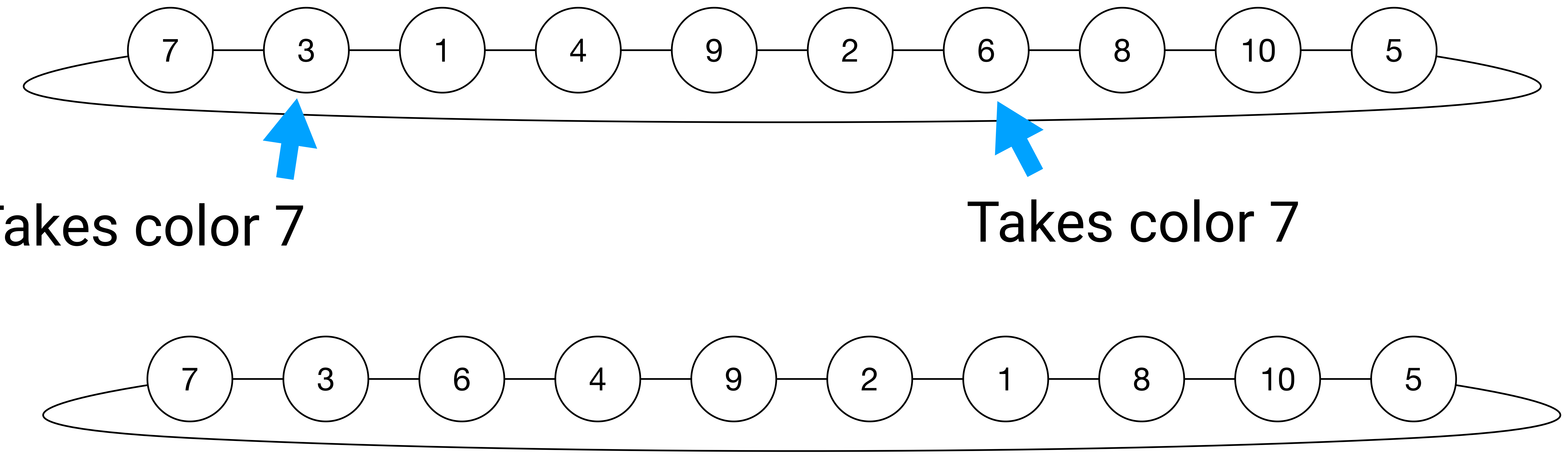

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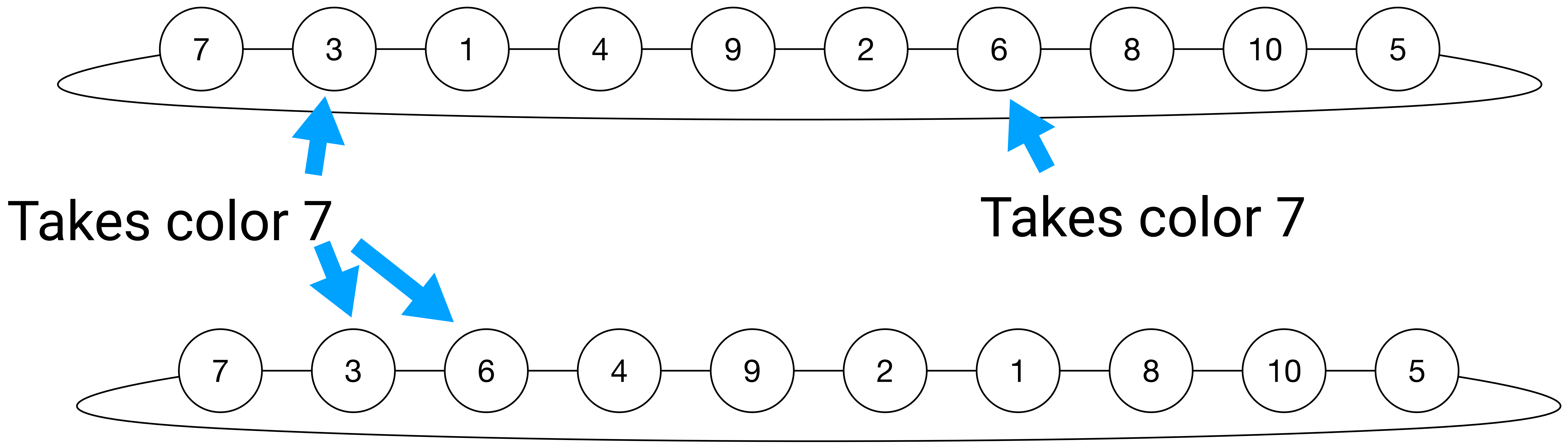

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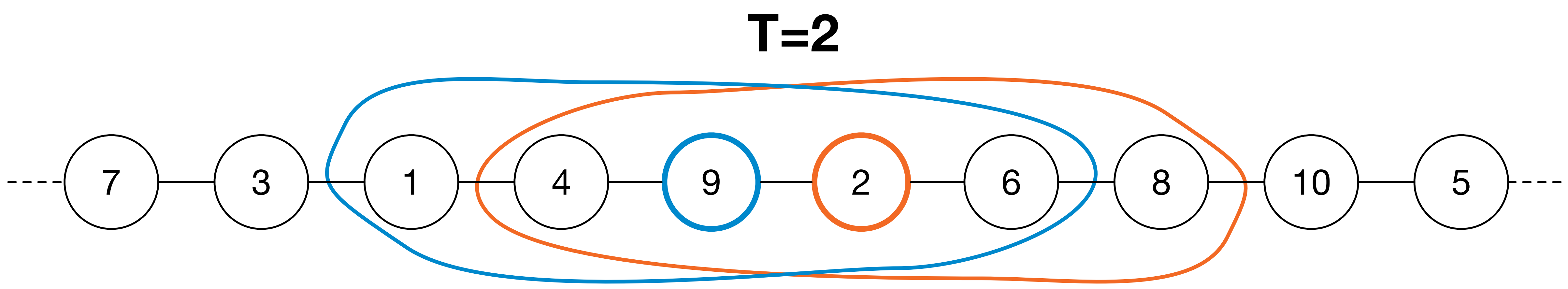

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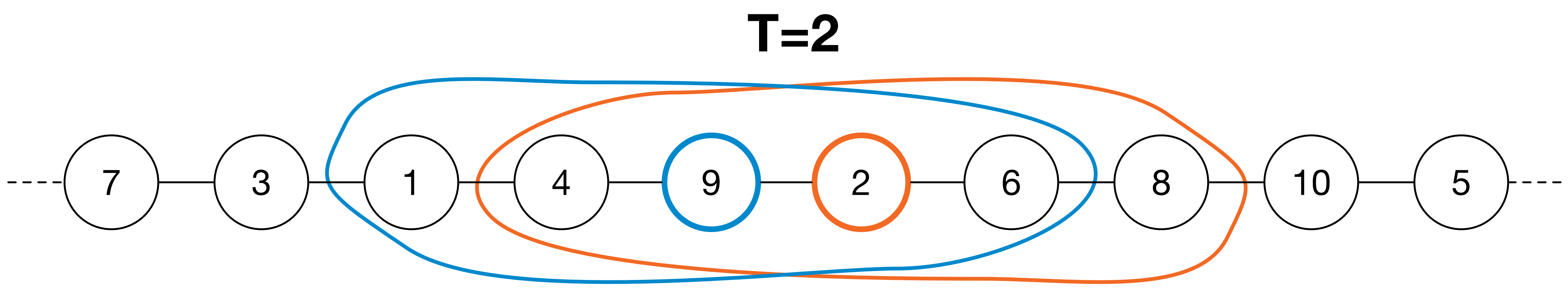

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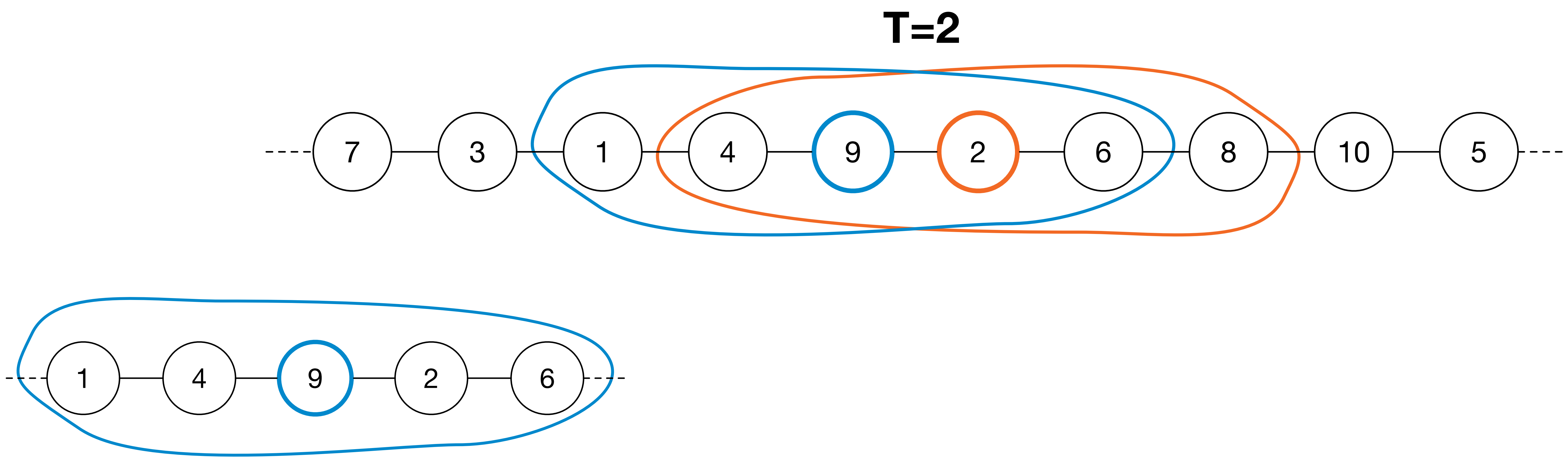

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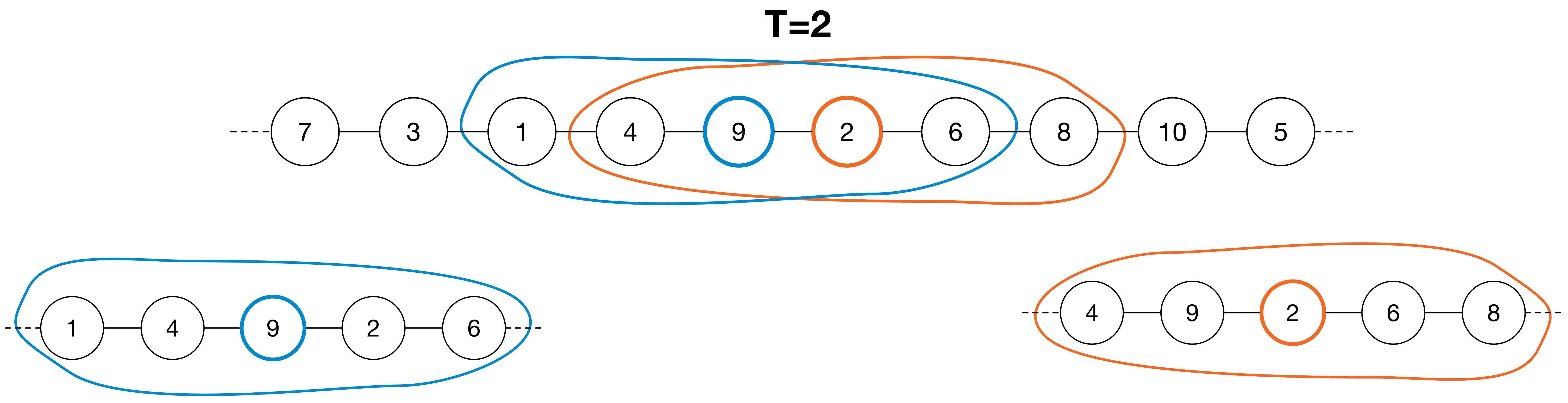

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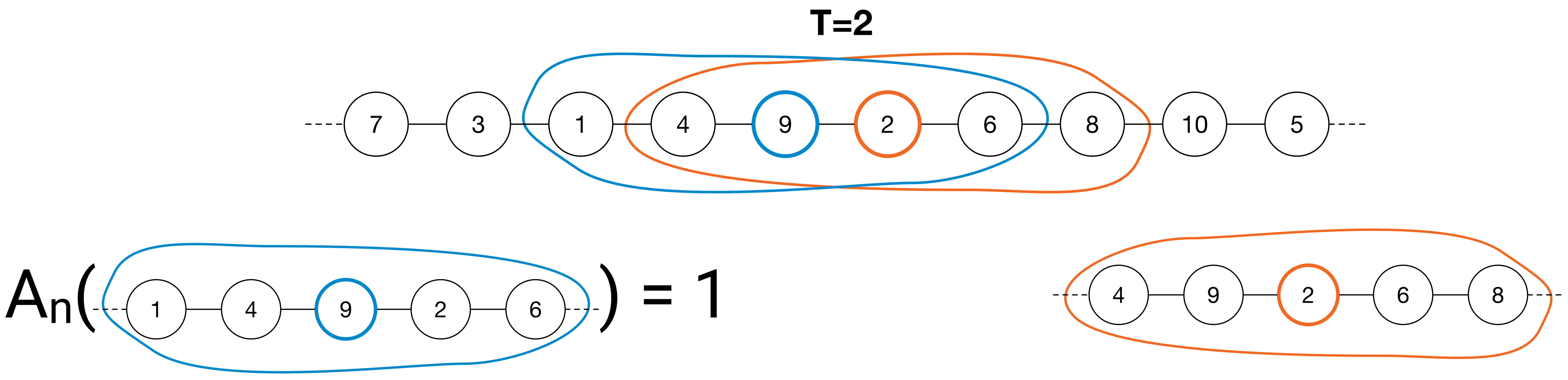

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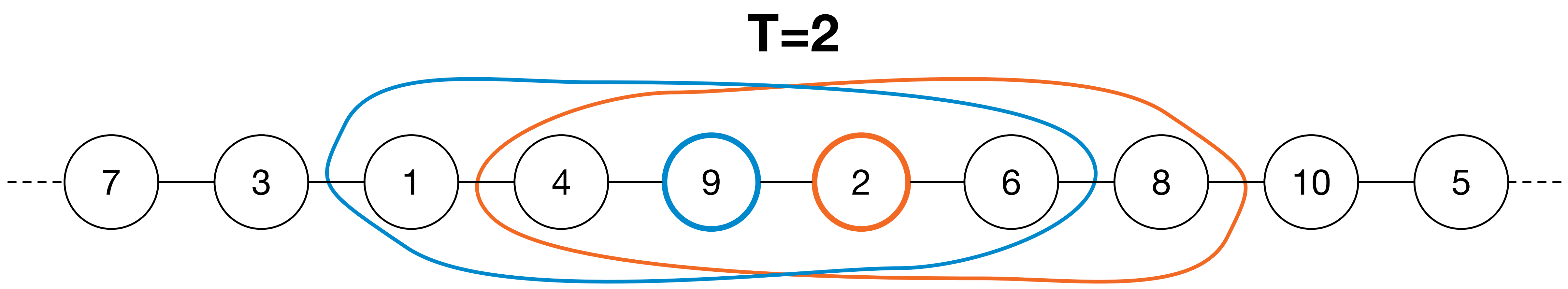

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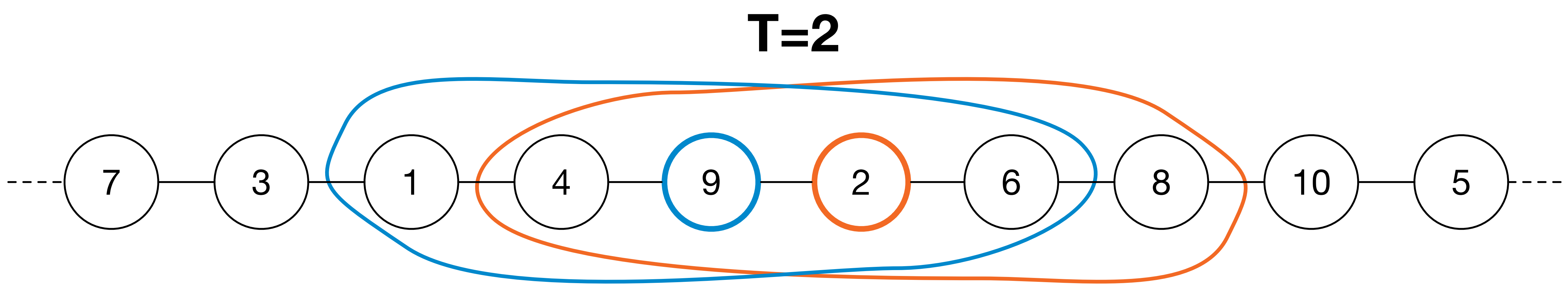
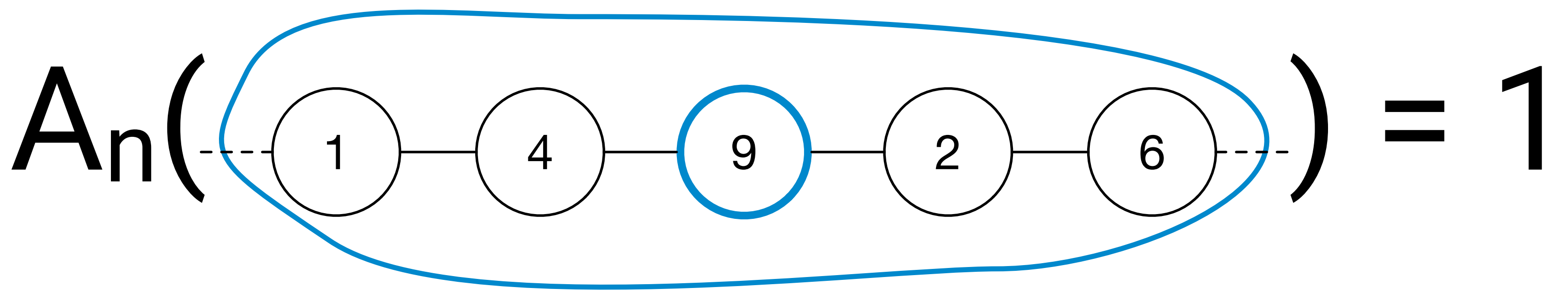
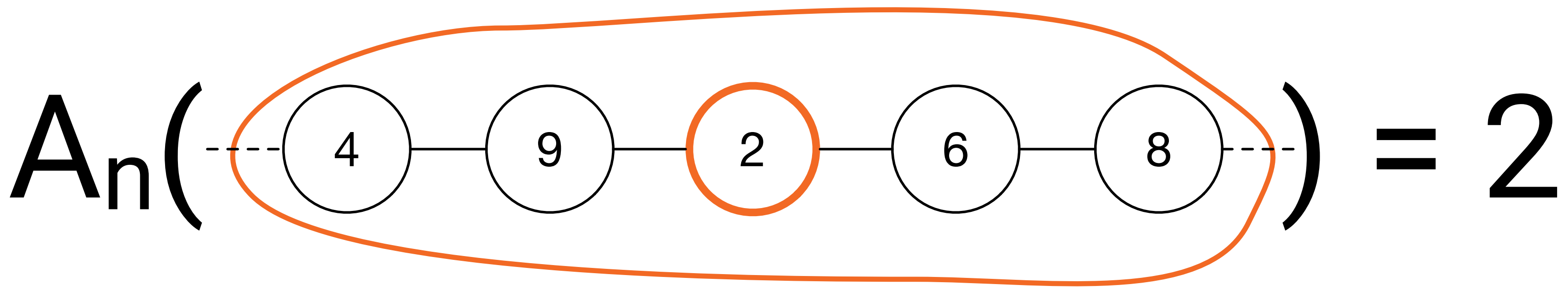
$$A_n(x_1, \ldots, x_{2T+1}) \neq A_n(x_2, \ldots, x_{2T+2})$$

assuming $x_1, \ldots, x_{2T+2}$ are all distinct numbers from $\{1, \ldots, n\}$







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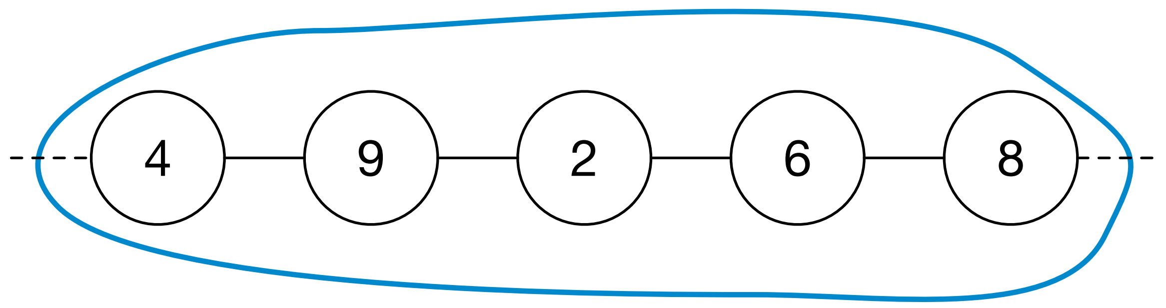

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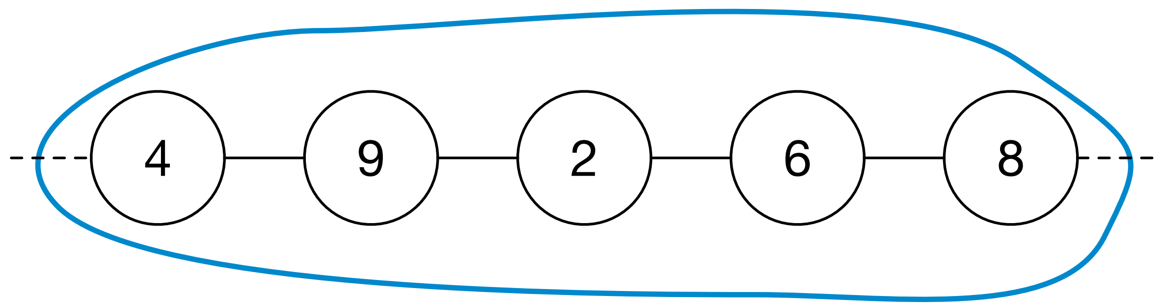

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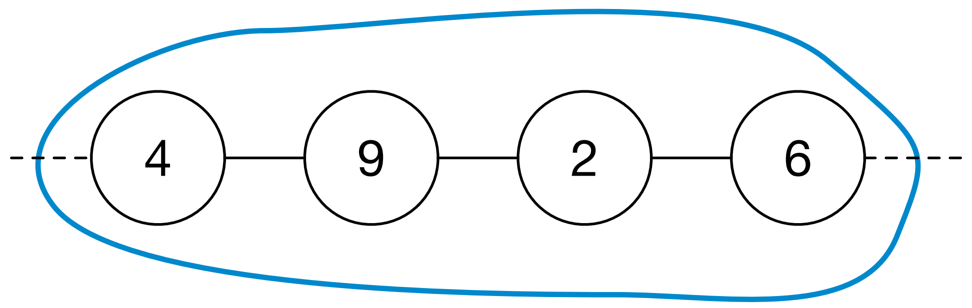

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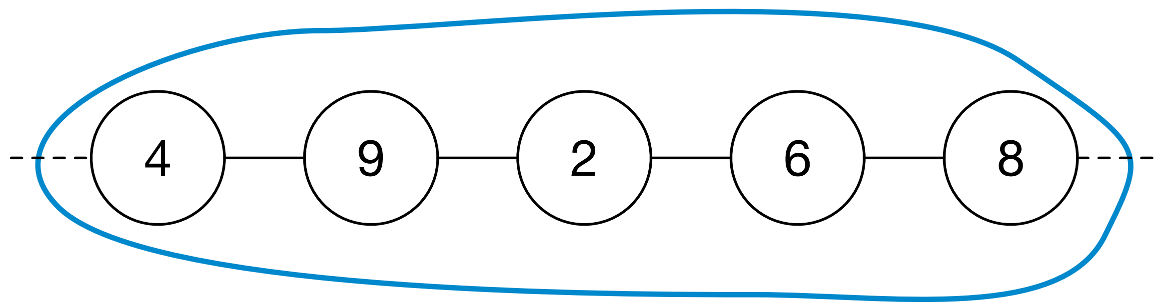

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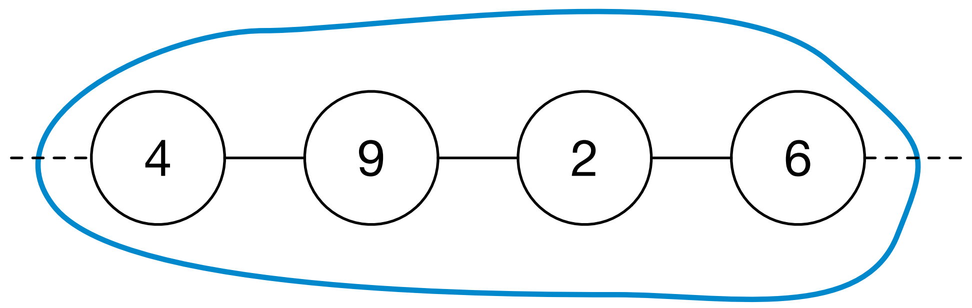

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4-ary 10-coloring function

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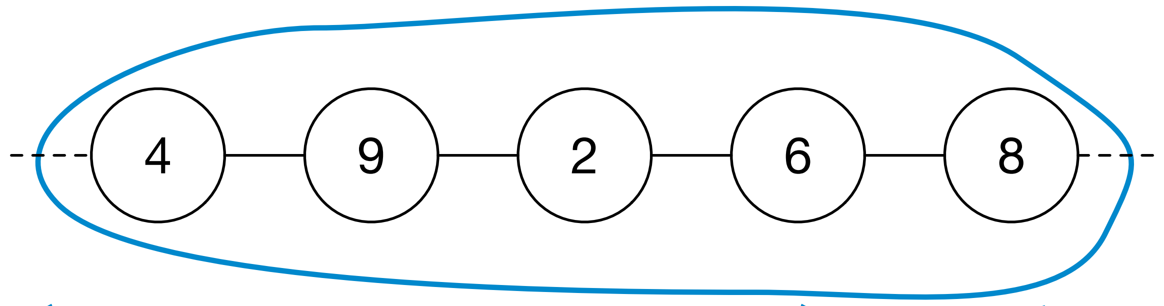

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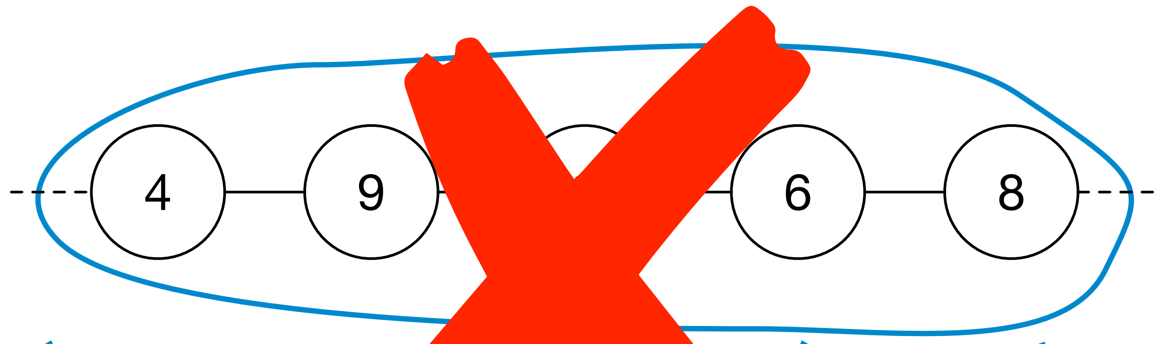

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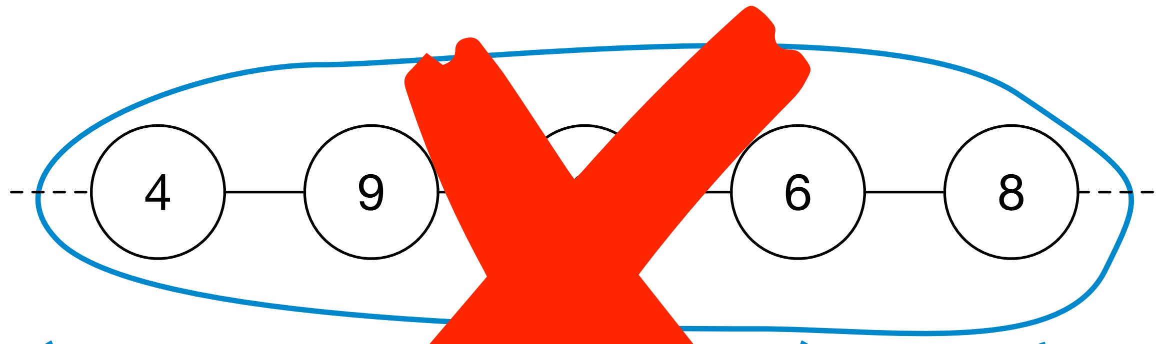

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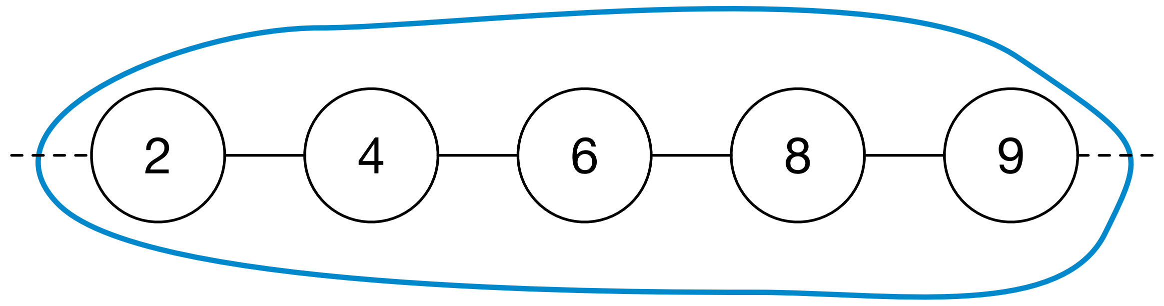

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We cannot use a function to construct an algorithm.

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- We prove this statement by induction

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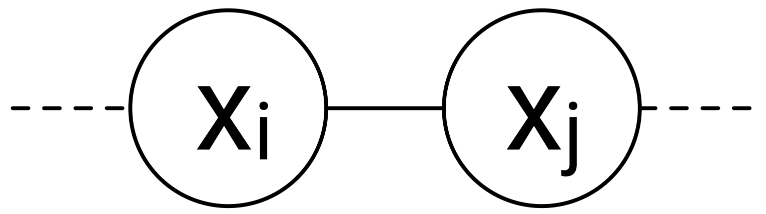

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- $A_n(x_1, \ldots, x_k) = A_n(x_2, \ldots, x_k, y)$, such that $y > x_k$, contradiction!

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If c-coloring can be solved in T rounds,

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Given a problem P_i, satisfying that the correctness of the solution can be checked locally, the problem P_{i+1} can be defined mechanically [Brandt '19]