Exercise 1: Proof by Induction  

Prove the following summation formulas for all \( n \in \mathbb{N} \) (i.e., all integers bigger 0) by induction on \( n \).

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

Exercise 2: Relations between Sets  

State whether the following set relations are true or false. Prove your claim.

(a) \( A \cup B \supseteq A \cap B \)

(b) \( A \cup B \subseteq A \cap B \)

(c) \( \{ n^3 + 2n \mid n \in \mathbb{N} \} \supseteq \{ n \mid n \in \mathbb{N} \text{ is divisible by 3} \} \)

(d) \( \{ n^3 + 2n \mid n \in \mathbb{N} \} \subseteq \{ n \mid n \in \mathbb{N} \text{ is divisible by 3} \} \)

Remark: Assume \( A, B \subseteq \Omega \) are two subsets from a groundset \( \Omega \) (the set of “all” elements). To disprove a relation from above it is sufficient to give concrete examples for \( A, B, \Omega \) for which the relation does not hold (i.e., a counterexample).

Exercise 3: Counting Edges in Acyclic Graphs  

A tree is an acyclic, connected, simple graph. Show by induction that a tree with \( n \geq 1 \) nodes has \( n-1 \) edges. A forest is an acyclic, simple graph (not necessarily connected, i.e., it may consist of several unconnected trees). Show that a forest consisting of \( k \) connected components has \( n-k \) edges.

Remark: Acyclic means that there are no cycles. A simple graph has no self-loops or multi-edges.

Exercise 4: Nodes with Identical Degrees  

Show that every simple graph with two or more nodes contains two nodes with the same degree.

Hint: A proof by contradiction is useful.