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# Theoretical Computer Science - Bridging Course
# Exercise Sheet 8

**Due:** Wednesday, 23rd of June 2021, 12:00 pm

## Exercise 1: Questions about class $\mathcal{NP}$ *(6 Points)*

- Give two equivalent definitions of the class $\mathcal{NP}$.
- Is the 3-Clique problem from the previous sheet in class $\mathcal{NP}$ ? Is it decidable? Justify.
- Is the halting problem in $\mathcal{NP}$? Justify.

## Exercise 2: Proving Problems in $\mathcal{NP}$ *(9 Points)*

Show that the following problems are in the class $\mathcal{NP}$

- A *clique* in a graph $G = (V, E)$ is a set $Q \subseteq V$ such that for all $u, v \in Q : \{u, v\} \in E$.
  Given a graph $G$ and integer $k$, it is required to determine whether $G$ contains a clique of size at least $k$, hence consider the following problem:
  Clique:= $\{\langle G, k\rangle \mid G$ has a clique of size at least $k$ $\}$.
- Given a collection $S$ of integers $x_1, \ldots, x_k$ and a target $t$, it is required to determine whether $S$ contains a sub-collection that adds up to $t$, hence consider the following problem:

$$\text{SUBSET-SUM} = \left\{\langle S, t\rangle | S = \{x_1, \ldots, x_k\}, \text{and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\} \text{ we have } \sum_i y_i = t\right\}$$

- A *hitting set* $H \subseteq \mathcal{U}$ for a given universe $\mathcal{U}$ and a set $S = \{S_1, S_2, \ldots, S_m\}$ of subsets $S_i \subseteq \mathcal{U}$, fulfills the property $H \cap S_i \neq \emptyset$ for $1 \leq i \leq m$ ($H$ 'hits' at least one element of every $S_i$).

  Given a universe set $\mathcal{U}$, a set $S$ of subsets of $\mathcal{U}$, and a positive integer $k$, it is required to determine whether $\mathcal{U}$ contains a hitting set of size at most $k$, hence consider the following problem: HittingSet := $\{\langle \mathcal{U}, S, k\rangle \mid$ universe $\mathcal{U}$ has subset of size $\leq k$ that *hits* all sets in $S \subseteq 2^{\mathcal{U}}\}$.[1]

## Exercise 3: From $\mathcal{NP}$ to $\mathcal{NPC}$ *(5 Points)*

A language is called $\mathcal{NP}$-complete ($\Leftrightarrow$: $L \in \mathcal{NPC}$), if

1. $L \in \mathcal{NP}$ and
2. $L$ is $\mathcal{NP}$-hard.

*Recall* what are $\mathcal{NP}$-hard problems and how to show a problem $\mathcal{NP}$-hard in the following.

[1]The power set $2^{\mathcal{U}}$ of some ground set $\mathcal{U}$ is the set of *all subsets* of $\mathcal{U}$. So $S \subseteq 2^{\mathcal{U}}$ is a collection of subsets of $\mathcal{U}$.

- Language $L$ is called $\mathcal{NP}$-hard, if *all* languages $L' \in \mathcal{NP}$ are polynomially reducible to $L$, i.e.

$$L \text{ is } \mathcal{NP}\text{-hard} \iff \forall L' \in \mathcal{NP} : L' \leq_p L.$$

  *Recall* let $L_1, L_2$ be languages (problems) over alphabets $\Sigma_1, \Sigma_2$. Then $L_1 \leq_p L_2$ ($L_1$ is polynomially reducible to $L_2$), iff a function $f : \Sigma_1^* \to \Sigma_2^*$ exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1^* : s \in L_1 \iff f(s) \in L_2.$$

- The reduction relation '$\leq_p$' is transitive ($L_1 \leq_p L_2$ and $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$). Therefore, in order to show that $L$ is $\mathcal{NP}$-hard, it suffices to reduce a known $\mathcal{NP}$-hard problem $\tilde{L}$ to $L$, i.e. $\tilde{L} \leq_p L$.

Given a graph $G = (V, E)$, an independent set of size $k$ of $G = (V, E)$ is a subset $I \subseteq V$ of nodes, such that $|I| = k$ and $\{u, v\} \notin E$ for all $u, v \in I$.
Show that IndependentSet $:= \{\langle G, k\rangle \mid G$ is a simple graph and has an independent set of size at least $k\}$ is in $\mathcal{NPC}$. Use that the Clique problem from the above exercise is $\in \mathcal{NPC}$.

*Hint: For the poly. transformation ($\leq_p$) you have to describe an algorithm (with poly. run-time!) that transforms an instance $\langle G, k\rangle$ of* Clique *into an instance $\langle G', k'\rangle$ of* IndependentSet *s.t. a clique of size at least $k$ in $G$ becomes a independent set of $G'$ of size at least $k'$ and vice versa!.*