Theoretical Computer Science - Bridging Course
Exercise Sheet 9
Due: Wednesday, 30th of June 2021, 12:00 noon

Exercise 1: More Problems in \( \mathcal{NP} \)  

(10 Points)

(a) Show \( \text{HALFCLIQUE} := \{ \langle G \rangle \mid \text{Graph } G \text{ with } n \text{ nodes has clique of size at least } \lceil n/2 \rceil \} \in \mathcal{NP} \).

Use that \( \text{CLIQUE} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has clique of size at least } k \} \in \mathcal{NP} \).

Hint: Describe an algorithm (with poly. run-time!) that transforms \( G \) and \( k \) into a graph \( G' \) by adding nodes and connecting them with edges in a suitable manner, s.t. a Clique of size \( k \) in \( G \) becomes a Clique of size \( \lceil n/2 \rceil \) in \( G' \) and vice versa(!).

(b) Show \( \text{DOMINATINGSET} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has a dominating set of size at most } k \} \in \mathcal{NP} \).

Use that \( \text{VERTEXCOVER} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k \} \in \mathcal{NP} \).

Remark: A dominating set is a subset of nodes of \( G \) such that every node not in the subset is adjacent to some node in the subset. A vertex cover is a subset of nodes of \( G \) such that every edge of \( G \) is incident to a node in the subset.

Hint: Transform a Graph \( G \) into a Graph \( G' \) such that a vertex cover of \( G \) will result in a dominating set \( G' \) and vice versa(!). Note that a dominating set is not necessarily a vertex cover (\( G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}) \) has the dominating set \( \{v_1, v_4\} \) which is not a vertex cover). Also a vertex cover is not necessarily a dominating set (consider isolated notes).

Exercise 2: Complexity Classes – Big Picture  

(10 Points)

(a) Why is \( \mathcal{P} \subseteq \mathcal{NP} \)?

(b) Why are \( \Sigma^*, \emptyset \notin \mathcal{NP} \)?

(c) Show that \( \mathcal{P} \cap \mathcal{NP} = \emptyset \) if \( \mathcal{P} \neq \mathcal{NP} \).

(d) We already know for the Halting Problem \( H \notin \mathcal{NP} \) from the previous exercise sheet. Show that \( H \) is \( \mathcal{NP} \)-hard. You can use that the following problem is \( \mathcal{NP} \)-hard

\[
\text{SAT} = \{ \langle \phi \rangle \mid \text{bool. formula } \phi \text{ has assignment of variables s.t. } \phi \text{ evaluates to TRUE.} \}
\]

Hint: For any boolean formula \( \phi \) give an algorithm \( A \) that stops if and only if \( \phi \) is satisfiable.

(e) Give two Venn Diagrams with sets \( \{ \Sigma^*, \emptyset \}, \mathcal{P}, \mathcal{NP}, \mathcal{NP} \} \text{-hard for } \mathcal{P} \neq \mathcal{NP} \text{ and } \mathcal{P} = \mathcal{NP} \).