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Theoretical Computer Science - Bridging Course
Exercise Sheet 9

Due: Wednesday, 30th of June 2021, 12:00 noon

Exercise 1: More Problems in $\mathcal{NPC}$ *(10 Points)*

(a) Show $\textsc{HalfClique} := \{\langle G\rangle \mid \text{Graph } G \text{ with } n \text{ nodes has clique of size at least } \lceil n/2\rceil\} \in \mathcal{NPC}$.

Use that $\textsc{Clique} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has clique of size at least } k\} \in \mathcal{NPC}$.

Hint: Describe an algorithm (with poly. run-time!) that transforms G and k into a graph G' by adding nodes and connecting them with edges in a suitable manner, s.t. a Clique of size k in G becomes a Clique of size $\lceil n/2\rceil$ in G' and vice versa(!).

(b) Show $\textsc{DominatingSet} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has a dominating set of size at most } k\} \in \mathcal{NPC}$.

Use that $\textsc{VertexCover} := \{\langle G, k\rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k\} \in \mathcal{NPC}$.

*Remark: A **dominating set** is a subset of nodes of G such that every node not in the subset is adjacent to some node in the subset. A **vertex cover** is a subset of nodes of G such that every edge of G is incident to a node in the subset.*

Hint: Transform a Graph G into a Graph G' such that a vertex cover of G will result in a dominating set G' and vice versa(!). Note that a dominating set is not necessarily a vertex cover ($G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$) has the dominating set $\{v_1, v_4\}$ which is not a vertex cover). Also a vertex cover is not necessarily a dominating set (consider isolated notes).

Exercise 2: Complexity Classes – Big Picture *(10 Points)*

(a) Why is $\mathcal{P} \subseteq \mathcal{NP}$?

(b) Why are $\Sigma^*, \emptyset \notin \mathcal{NPC}$?

(c) Show that $\mathcal{P} \cap \mathcal{NPC} = \emptyset$ if $\mathcal{P} \neq \mathcal{NP}$.

(d) We already know for the Halting Problem $H \notin \mathcal{NP}$ from the previous exercise sheet. Show that H is $\mathcal{NP}$-hard. You can use that the following problem is $\mathcal{NP}$-hard

$$\text{SAT} = \{\langle \phi\rangle \mid \text{bool. formula } \phi \text{ has assignment of variables s.t. } \phi \text{ evaluates to TRUE.}\}$$

Hint: For any boolean formula ϕ give an algorithm $\mathcal{A}$ that stops if and only if ϕ is satisfiable.

(e) Give two Venn Diagrams with sets $\{\Sigma^*, \emptyset\}, \mathcal{P}, \mathcal{NP}, \mathcal{NPC}, \mathcal{NP}$-hard for $\mathcal{P} \neq \mathcal{NP}$ and $\mathcal{P} = \mathcal{NP}$.