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Theoretical Computer Science - Bridging Course
Exercise Sheet 10

Due: Wednesday, 7th of July 2021, 12:00 pm

Exercise 1: Is it a Tautology? *(4 Points)*

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are tautologies?

(a) $\varphi_1 = (p \land \neg q) \lor (\neg p \lor q)$

(b) $\varphi_2 = (\neg p \land (\neg p \lor q)) \leftrightarrow (p \lor \neg q)$

(c) $\varphi_3 = (p \land \neg q) \to \neg(p \land q)$

Remark: $a \to b :\equiv \neg a \lor b$, $a \leftrightarrow b :\equiv (a \to b) \land (b \to a), a \not\to b :\equiv \neg(a \to b)$.

Exercise 2: Logical: Equivalency and Entailment *(8 Points)*

- Two logical formulae φ, ψ over a set of atoms Σ are logically equivalent ($\varphi \equiv \psi$) iff for all interpretations I of Σ the following holds

$$I \models \varphi \Longleftrightarrow I \models \psi.$$

  With the above definition, show or disprove the following equivalencies (e.g. by making truth tables).

  (a) $\neg(p \to q) \equiv p \to \neg q$

  (b) $\neg(p \land q) \equiv \neg p \lor \neg q$

- A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

  Let $KB := \{p \lor (q \land \neg r), \neg r \land p\}$. Show or disprove that KB logically entails the following formulae.

  (a) $\varphi_1 := (p \land q) \lor \neg(\neg r \lor p)$

  (b) $\varphi_2 := (q \leftrightarrow r) \to p$

Exercise 3: CNF, DNF *(4 Points)*

(a) Convert

$$\varphi_1 := (\neg p \to q) \to (q \to \neg r)$$

into Conjunctive Normal Form (CNF).

(b) Convert

$$\varphi_2 := \neg(p \to q) \lor ((r \lor s) \to (q \lor t)) \lor (\neg p \to \neg v)$$

into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

Exercise 4: Inference Rules and Calculi *(4 Points)*

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* $\mathbf{C}$ is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus $\mathbf{C}$ to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \land \psi}{\varphi, \psi}, \frac{(\varphi \land \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \land q, p \to r, (q \land r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.