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Theoretical Computer Science - Bridging Course
Exercise Sheet 11

Due: Wednesday, 7th of June 2021, 12:00 noon

Exercise 1: Completeness and Correctness of Calculi *(10 Points)*

A calculus $\mathbf{C}$ is called *correct* if for every knowledge base KB and formula φ the following holds

$$KB \vdash_{\mathbf{C}} \varphi \quad \Longrightarrow \quad KB \models \varphi.$$

Calculus $\mathbf{C}$ is called *complete* if

$$KB \models \varphi \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \varphi.$$

Consider the following calculi

$$\mathbf{C_1}: \quad \frac{\varphi \leftrightarrow \psi}{\varphi \rightarrow \psi, \psi \rightarrow \varphi} \qquad \mathbf{C_2}: \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi} \qquad \mathbf{C_3}: \quad \frac{\varphi, \psi \rightarrow \varphi}{\psi}$$

(a) Show that the $\mathbf{C_1}$ and $\mathbf{C_2}$ are both correct by using truth tables and/or giving a short explanation.

(b) Show that $\mathbf{C_3}$ is not correct.

(c) Show that $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C_3}$ are not complete by giving a knowledge base KB and a formula φ such that $KB \models \varphi$ but not $KB \vdash_{\mathbf{C_i}} \varphi$.

Exercise 2: Resolution Calculus *(10 Points)*

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

In order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus $\mathbf{C}$ is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Hence, given a refutation-complete calculus $\mathbf{C}$ it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ to prove $KB \models \varphi$.

The *Resolution Calculus*[1] $\mathbf{R}$ is correct and refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$
Remark: KB represents the formula $C_1 \wedge \ldots \wedge C_n$ with $C_i = L_{i,1} \vee \ldots \vee L_{i,m_i}$.

[1] Complete calculi are impractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L, \neg L\} \vdash_{\mathbf{R}} \bot$).

(a) We want to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$.

(c) Consider the sentence "Heads, I win". "Tails, you lose". Design a propositional KB that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that **I always win**.