Exercise 1: Completeness and Correctness of Calculi \textit{(10 Points)}

A calculus $\mathcal{C}$ is called \textit{correct} if for every knowledge base $KB$ and formula $\varphi$ the following holds

\[ KB \vdash_{\mathcal{C}} \varphi \implies KB \models \varphi. \]

Calculus $\mathcal{C}$ is called \textit{complete} if

\[ KB \models \varphi \implies KB \vdash_{\mathcal{C}} \varphi. \]

Consider the following calculi

- $\mathcal{C}_1: \frac{\varphi \leftrightarrow \psi}{\varphi \rightarrow \psi, \psi \rightarrow \varphi}$
- $\mathcal{C}_2: \frac{\varphi, \varphi \rightarrow \psi}{\psi}$
- $\mathcal{C}_3: \frac{\varphi, \psi \rightarrow \varphi}{\psi}$

(a) Show that the $\mathcal{C}_1$ and $\mathcal{C}_2$ are both correct by using truth tables and/or giving a short explanation.

(b) Show that $\mathcal{C}_3$ is not correct.

(c) Show that $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are not complete by giving a knowledge base $KB$ and a formula $\varphi$ such that $KB \models \varphi$ but not $KB \vdash_{\mathcal{C}_i} \varphi$.

Exercise 2: Resolution Calculus \textit{(10 Points)}

Due to the \textit{Contradiction Theorem} (cf. lecture) for every knowledge base $KB$ and formula $\varphi$ it holds

\[ KB \models \varphi \iff KB \cup \{\neg \varphi\} \models \bot. \]

Remark: $\bot$ is a formula that is unsatisfiable.

In order to show that $KB$ entails $\varphi$, we show that $KB \cup \{\neg \varphi\}$ entails a contradiction. A calculus $\mathcal{C}$ is called \textit{refutation-complete} if for every knowledge base $KB$

\[ KB \models \bot \implies KB \vdash_{\mathcal{C}} \bot. \]

Hence, given a refutation-complete calculus $\mathcal{C}$ it suffices to show $KB \cup \{\neg \varphi\} \vdash_{\mathcal{C}} \bot$ to prove $KB \models \varphi$.

The \textit{Resolution Calculus} is correct and refutation-complete for knowledge bases that are given in \textit{Conjunctive Normal Form} (CNF). A knowledge base $KB$ is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of $m_i$ literals $L_{i,j}$

Remark: $KB$ represents the formula $C_1 \land \ldots \land C_n$ with $C_i = L_{i,1} \lor \ldots \lor L_{i,m_i}$.

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\textsuperscript{1}Complete calculi are impractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.
The Resolution Calculus has only one inference rule, the resolution rule:

\[ \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2} \]

Remark: \(L\) is a literal and \(C_1 \cup \{L\}, C_2 \cup \{\neg L\}\) are clauses in \(KB\) (\(C_1, C_2\) may be empty). To show \(KB \vdash R \bot\), you need to apply the resolution rule, until you obtain two conflicting one-literal clauses \(L\) and \(\neg L\). These entail the empty clause (defined as \(\Box\)), i.e. a contradiction (\(\{L, \neg L\} \vdash R \bot\)).

(a) We want to show \(\{p \land q, p \rightarrow r, (q \land r) \rightarrow u\} \models u\). First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show \(\{p \land q, p \rightarrow r, (q \land r) \rightarrow u\} \models u\).

(c) Consider the sentence “Heads, I win”. “Tails, you lose”. Design a propositional \(KB\) that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that I always win.