University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
S. Faour, A. Mályusz

Theory of Distributed Systems
Sample Solution Exercise Sheet 1

Due: Wednesday, 24th of April 2024, 12:00 noon

Exercise 1: Schedules and Correctness Properties *(5 Points)*

(a) Consider three nodes, v_1, v_2, and v_3, which are connected via FIFO channels, that is, messages between any two nodes are received in the same order they are sent. For example, if node v_1 sends first message m_1 then m_2 to node v_2, then v_2 will first receive m_1 and then m_2.

Devise **one** possible schedule S which is consistent with the following local restrictions to the three nodes.

- $S|1 = s_{1,3}\ s_{1,3}\ r_{1,2}\ r_{1,3}\ s_{1,2}\ r_{1,2}\ s_{1,3}$,
- $S|2 = s_{2,3}\ s_{2,1}\ r_{2,1}\ s_{2,1}$,
- $S|3 = r_{3,2}\ r_{3,1}\ s_{3,1}\ r_{3,1}\ r_{3,1}$.

$s_{i,j}$ denotes the send event from node i to node j and $r_{j,i}$ denotes the event that node j receives a message from node i.

(b) Consider that the communication service above ensures that messages exchanged between the nodes are:

- never lost
- never duplicated
- received in the same order they are sent

Specify for each which correctness property it represents.

Sample Solution

(a) There could be more than one possible global schedule S. Two possible ones are the following.

$$s_{2,3}\ s_{1,3}\ r_{3,2}\ r_{3,1}\ s_{2,1}\ s_{1,3}\ s_{3,1}\ r_{1,2}\ r_{1,3}\ r_{3,1}\ s_{1,2}\ r_{2,1}\ s_{2,1}\ r_{1,2}\ s_{1,3}\ r_{3,1}.$$

$$s_{1,3}\ s_{2,3}\ r_{3,2}\ r_{3,1}\ s_{1,3}\ s_{2,1}\ s_{3,1}\ r_{3,1}\ r_{1,2}\ r_{1,3}\ s_{1,2}\ r_{2,1}\ s_{2,1}\ r_{1,2}\ s_{1,3}\ r_{3,1}.$$

One can also get a possible solution by drawing the graphical diagram as in the lecture.

(b) Messages are not lost is a liveness property and the remaining ones are a safety property. Note that we are not relying on a formal definition for the correctness propeties (one can find it outside the scope of this lecture), rather on the informal notions as stated in the lecture.

Exercise 2: The Level Algorithm *(5 Points)*

Consider the following algorithm between two connected nodes u and v:

The two nodes maintain levels ℓ_u and ℓ_v, which are both initialized to 0. One round of the algorithm works as follows:

1. Both nodes send their current level to each other
2. If u receives level ℓ_v from v, u updates its level to $\ell_u := \max\{\ell_u, \ell_v + 1\}$. If the message to node u is lost, node u does not change its level ℓ_u. Node v updates its level ℓ_v in the same (symmetric) way.

Argue that if the level algorithm runs for r rounds, the following properties hold:

(a) At the end, the two levels differ by at most one.

(b) If all messages succeed, both levels are equal to r.

(c) The level of a node is at least 1 if and only if the node received at least one message.

Sample Solution

a) We argue by induction on the number of rounds r that : after r rounds, the two levels differ by at most one. For a round r, let ℓ_u^r and ℓ_v^r be the levels of nodes u and v after round r. We have that $r \geq 0$. So for the base case: for $r = 0$ we have that $\ell_u^0 = 0 = \ell_v^0$. For the inductive step: assume that the statement holds after round r i.e. , we have $|\ell_u^r - \ell_v^r| \leq 1$, which is equivalent to $\ell_v^r - 1 \leq \ell_u^r \leq \ell_v^r + 1$. We have

$$\ell_u^{r+1} \leq \max\{\ell_u^r, \ell_v^r + 1\} \stackrel{\text{I.H.}}{=} \ell_v^r + 1 \leq \ell_v^{r+1} + 1$$

where the last inequation holds because levels can only increase.

Analogously, we prove $\ell_v^{r+1} \leq \ell_u^{r+1} + 1$.

b) Induction on the number of rounds: At the beginning (after round 0), we have $\ell_u^0 = \ell_v^0 = 0$. Now assume $\ell_u^r = \ell_v^r = r$ and in round r both messages succeed. Then $\ell_u^{r+1} = \max\{\ell_u^r, \ell_v^r + 1\} = \max\{r, r+1\} = r + 1$ and $\ell_v^{r+1} = \max\{\ell_v^r, \ell_u^r + 1\} = r + 1$.

c) For the forward direction: if a node never receives a message, it never updates its level (which is initially 0). So if its level is at least one, then it must have received a message.
For the backward direction: if node u receives $\ell_v \geq 0$ in some round, then its level becomes $\ell_v + 1 \geq 1$ which never decreases again.

Exercise 3: (Variations) of Two Generals *(10 Points)*

In the lecture we considered the (deterministically unsolvable) **Two Generals** consensus problem:

- two deterministic nodes, synchronuous communication, unreliable messages,
- **input**: 0 or 1 for each node,
- **output**: each node needs to decide either 0 or 1,
- **agreement**: both nodes must output the same decision (0 or 1),
- **validity**: if both nodes have the same input $x \in \{0, 1\}$ and no messages are lost, both nodes output x,

- **termination**: both nodes terminate in a bounded number of rounds.

(a) Generalize the problem to an arbitrary number $n \geq 2$ of generals i.e. whenever there are two nodes in the problem description above, we replace it with n nodes. Show that even with this generalization the problem remains deterministically unsolvable.

For the rest of the exercise, we consider three modifications of the model. For each of them, either give a (deterministic) algorithm or state a proof which shows that the variation cannot be solved deterministically.

(b) There is the guarantee that within the first 10 rounds at least *one* message in *each* direction succeeds.

(c) There is the guarantee that within the first 10 rounds at least *one* message succeeds.
Remark: *nodes are not allowed to stay silent.*

(d) Let $k \in \mathbb{N}$ be a natural number. The input for each node is a number $x \in \{0, \ldots, k\}$.

Goal: If no message gets lost *and* both have the same input $x \in \{0, \ldots, k\}$, both have to output x. In all other cases the nodes should output numbers which do not differ by more than one. The algorithm still has to terminate in a finite number of rounds.

Sample Solution

(a) Just like the 2 generals problem, the high level idea is to argue by contradiction and assume there is a deterministic algorithm that solves the generalized problem, then start with an execution in which all nodes have input 0 and no messages are lost, so after running the algorithm all nodes will decide 0. Next, for every pairwise nodes we do the same as the 2 generals problem where: we prune messages one by one to get a sequence of executions s.t. the consecutive executions are similar and then from an execution with no messages lost between these 2 nodes and both inputs 0, we can get to an execution with no messages delivered between these 2 nodes and both inputs are 1. Finally by adding back messages one-by-one between each pairwise node , we get to an execution in which "all" nodes have input 1 and no messages are lost, thus all nodes must output 1 but they all output 0, a contradiction. Thus the generalized version of the problem is also unsolvable.

(b) As there is the guarantee that at least one message in *each* direction succeeds every node sends its value repeatedly for ten rounds. At least one time it will reach the other node. So both nodes know both values and can decide on a output by a previously fixed algorithm, e.g., output $\min(val_1, val_2)$ or output $val_1 \cdot val_2$.

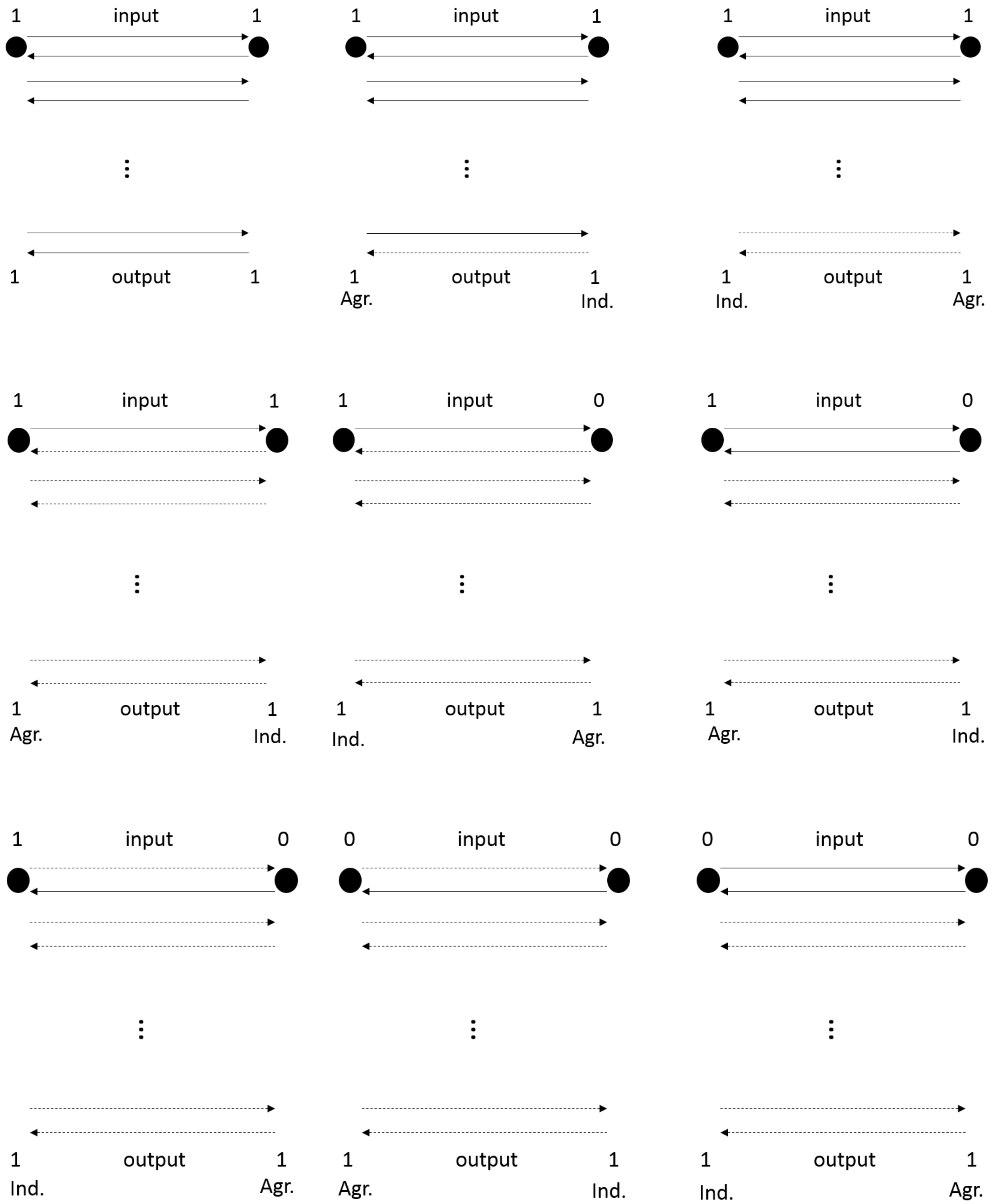
(c) The problem is not solvable for two deterministic generals. Assume that it is solvable in $T \geq 10$ rounds. By a sequence of executions, we show that both nodes need to have the same output in the following two executions.

- E: both inputs are 1, all messages are delivered
- E': both inputs are 0, all messages are delivered

Because of validity both nodes need to output 1 in the case of E and 0 in the case of E', a contradiction.

The proof is similar to the proof in the lecture - one only has to be careful in the last step to always keep at least one successfully delivered message (because of the guarantee in this problem's description).

Important is that we drop at most one message at a time to keep the two following executions similar (see definition of similar in the lecture). The reasoning is always that two similar executions are indistinguishable for one node and thus it has to output the same value in both execution. Then, to fulfill agreement all nodes need to output the same in both executions.



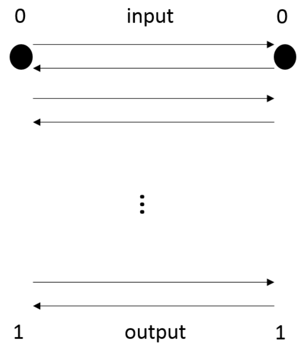


Figure 1: We begin with the execution E and show that it is indistinguishable from E' via a long chain of indistinguishable executions.

(d) The problem is solvable with the help of the level algorithm from exercise 2. Let x_u and x_v be respectively the input of node u and the input of node v. The algorithm is the following: each node runs the Level algorithm for as many rounds as the value they have in input, and then they output their level.

- Case 1: $x_u = x_v = x$. In this case, both nodes will run the Level algorithm for x many rounds. We distinguish two cases.
  (a) If no messages are lost, by exercise 2 b we know that both levels are equal to x, hence the output of both nodes will be x, as desired.
  (b) If there are lost messages, by exercise 2 a we know that the nodes' levels differ by at most one, hence their output will differ by at most one, as desired.
- Case 2: $x_u \neq x_v$. In this case, one of the nodes will run for more rounds than the other. Notice that this is analogous to the case where both nodes have the same input but some messages are lost, and for this case we proved above that the outputs differ by at most one, as desired.