Chapter 2
Broadcast, Convergecast, and Spanning Trees

Distributed Systems

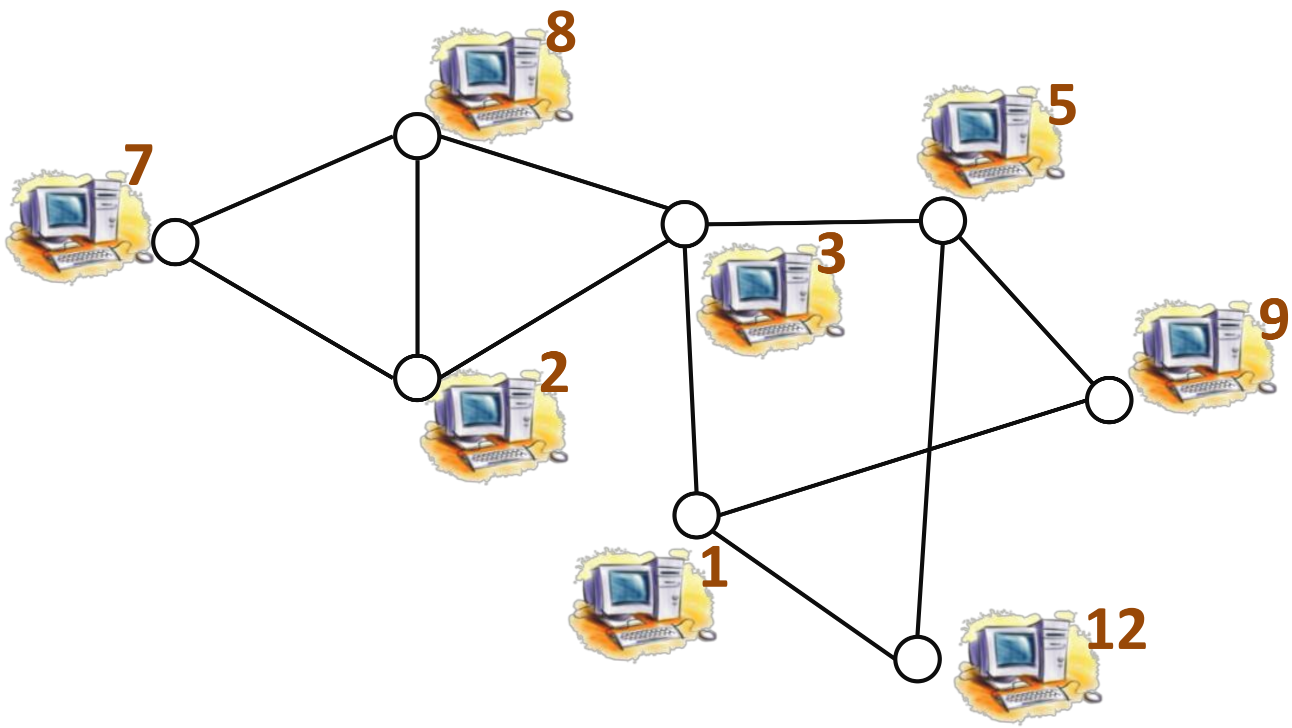
Summer Term 2024

Fabian Kuhn

Message Passing in Arbitrary Topologies

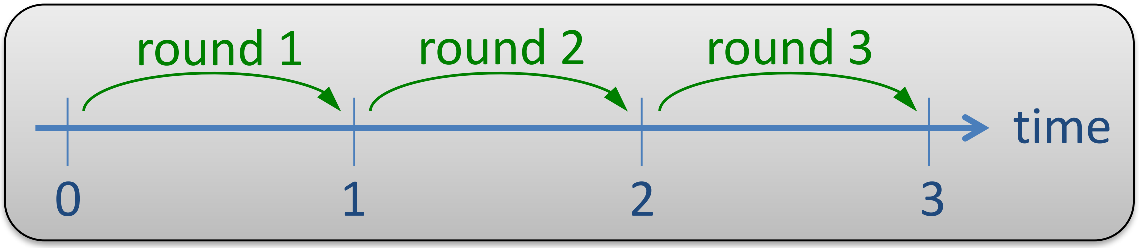
Assumption for this chapter:

- Network: message passing system with arbitrary topology
- network topology is given by an undirected graph $G = (V, E)$



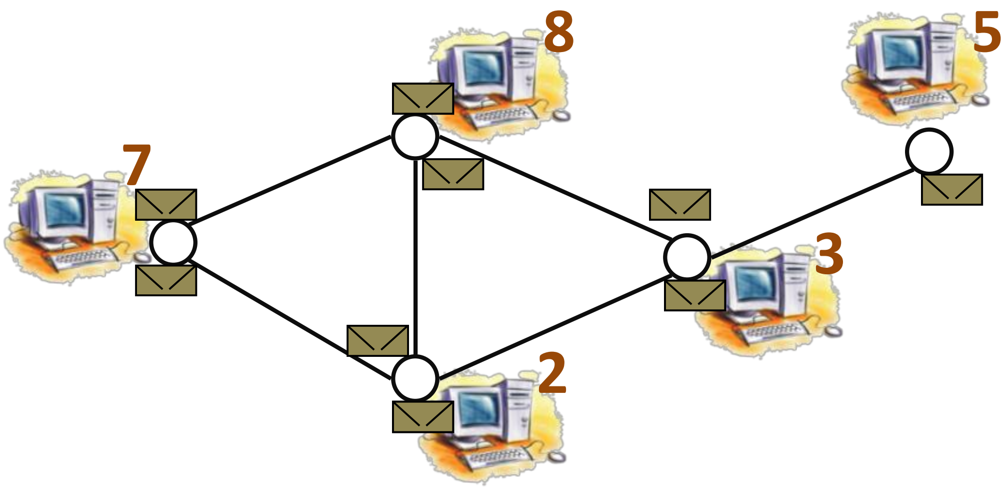
Synchronous Message Passing

- Time is divided into synchronous rounds



In each synchronous round:

1. Each node does some internal computation
2. Send a message to each neighbor
3. Receive message from each neighbor



time complexity = number of rounds

Asynchronous Message Passing

In this chapter: No failures, but asynchrony

Asynchronous message passing:

- messages are always delivered in finite time
  - cf.: finite time → admissible schedule
- message delays are completely unpredictable
- algorithms are **event-based**:

upon receiving message from neighbor ..., do

some local computation steps

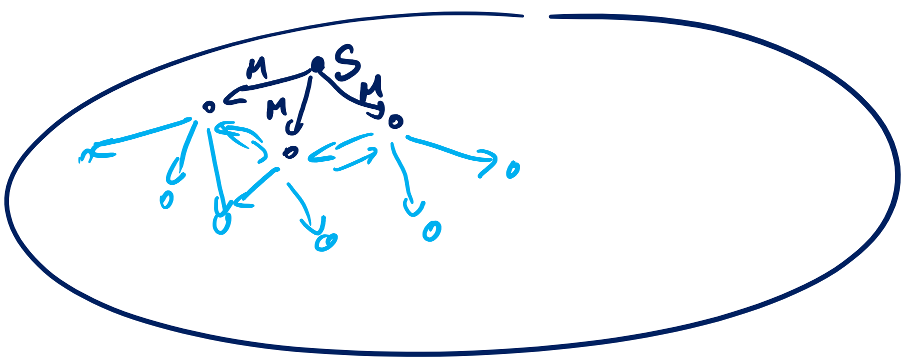
send message(s) to neighbor(s) ...

Broadcast

- Simple, basic communication problem

Problem Description:

- A source node s needs to broadcast a message M to all nodes of the system (network)
- Each node has a unique ID
- Initially, each node knows the IDs of its neighbors
  - or it can distinguish its neighbors by having individual communication ports to the pairwise communication links



Flooding

- One of the simplest distributed (network) algorithms

Basic idea:

- When receiving M for the first time, forward to all neighbors

Algorithm:

- Source node s:
  initially do
  send M to all neighbors

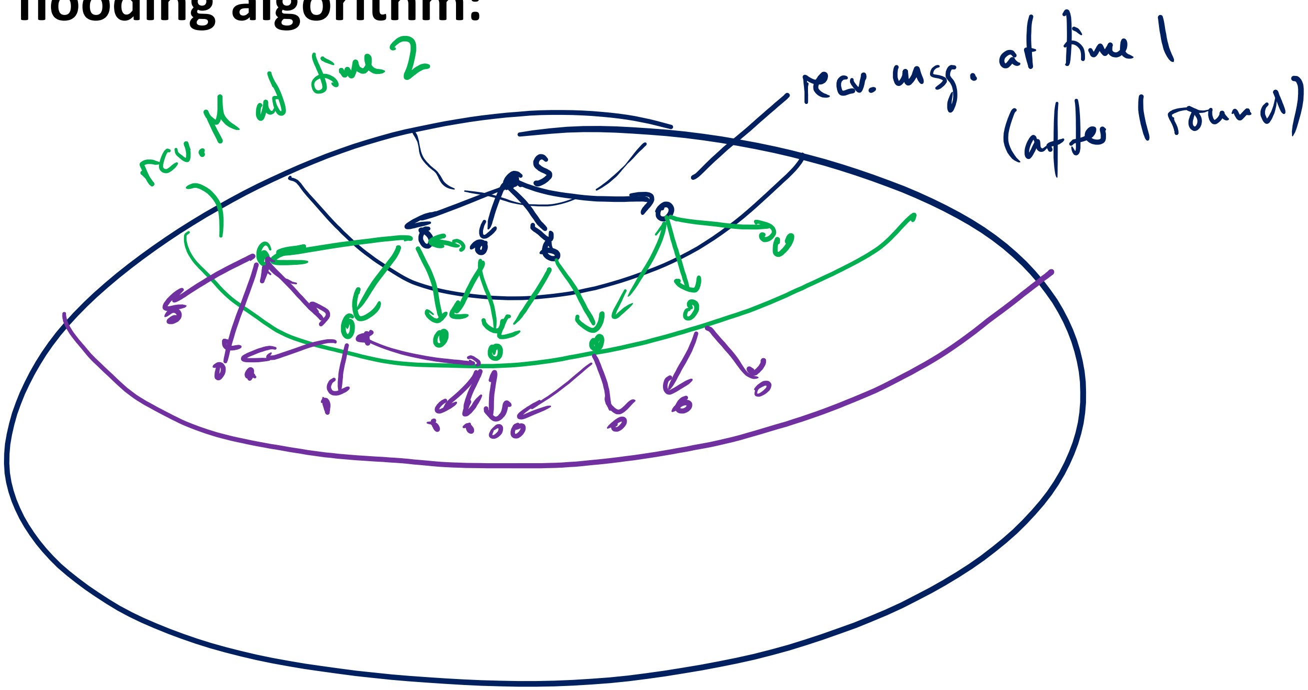
- All other nodes u:
  upon receiving M from some neighbor v for the first time
  if M has not been received before **then**
  send M to all neighbors except v

Flooding in Synchronous Systems

Synchronous systems:

- time divided into synchronous rounds, msg. delay = 1 round
- time complexity: number of rounds

Progress in flooding algorithm:



Flooding in Synchronous Systems

Synchronous systems:

- time divided into synchronous rounds, msg. delay $= 1$ round
- time complexity: number of rounds

Progress in flooding algorithm:

- after 1 round:
  - all neighbors of s know M
  - nodes at distance ≥ 2 from s do not know M
- after 2 rounds:
  - exactly nodes at distance ≤ 2 from s know M
- ...
- after r rounds:
  - exactly nodes at distance $\leq r$ from s know M

Flooding in Synchronous Systems

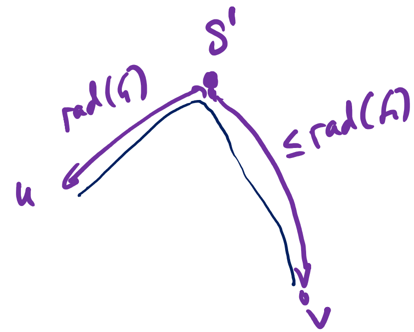
Radius: (Graph $G = (V, E)$)

- Given a node $s \in V$, radius of s in G:

$$rad(G, s) := \max_{v \in V} dist_G(s, v)$$

- radius of G:

$$rad(G) := \min_{s \in V} rad(G, s)$$



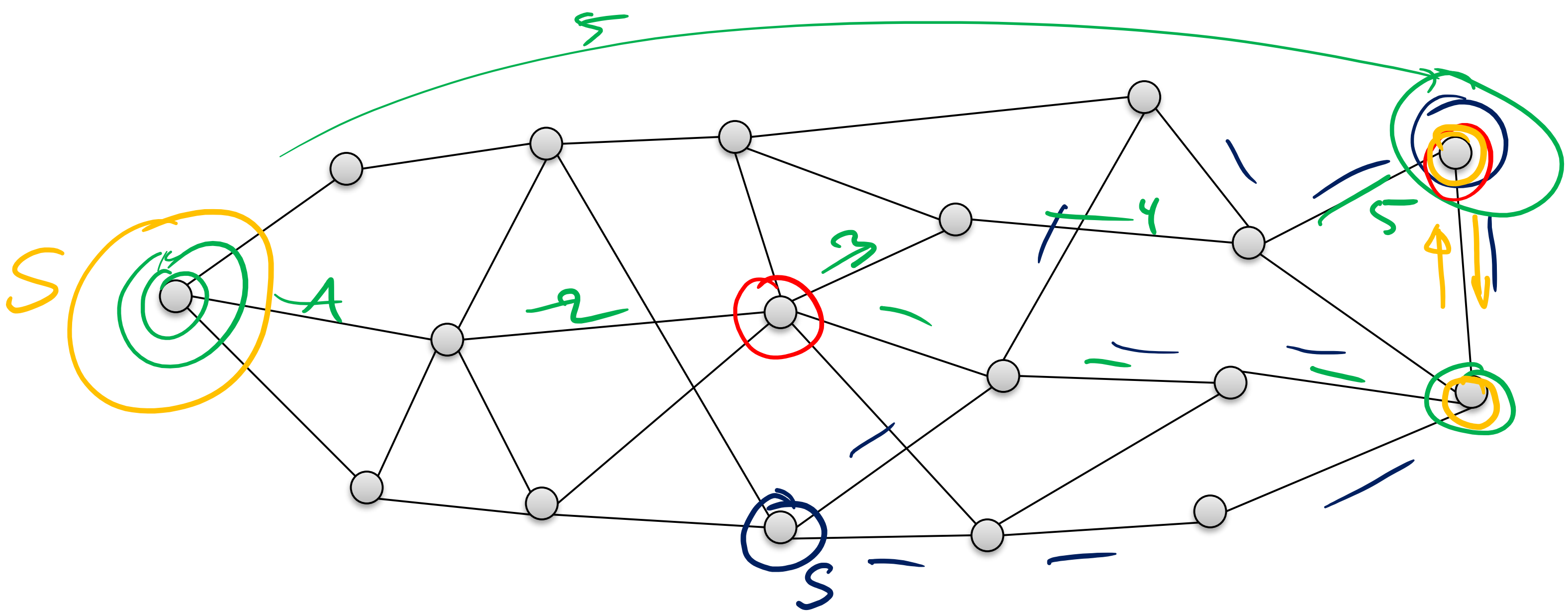
Diameter of G:

$$diam(G) := \max_{u,v \in V} dist_G(u, v) = \max_{s \in V} rad(G, s)$$

Time complexity of **flooding** in synchronous systems: $\boldsymbol{rad(G, s)}$

$$\frac{diam(G)}{2} \leq rad(G) \leq rad(G, s) \leq diam(G)$$

Radius and Diameter



$rad(G, s) = 4$

$diam(G) = 5$

$rad(G) = 3$

Asynchronous Time Complexity

- Time complexity of flooding in asynchronous systems?
- How is time complexity in asynchronous systems defined?

Assumptions:

- Message delays, time for local computations are arbitrary
  - Algorithms cannot use any timing assumptions!
- **For analysis:**
  - message delays ≤ 1 time unit
  - local computations take 0 time

Determine asynchronous time complexity:

1. determine running time of a given execution
2. **asynch. time complexity $=$ max. running time of any exec.**

Asynchronous Time Complexity

Running time of an execution:

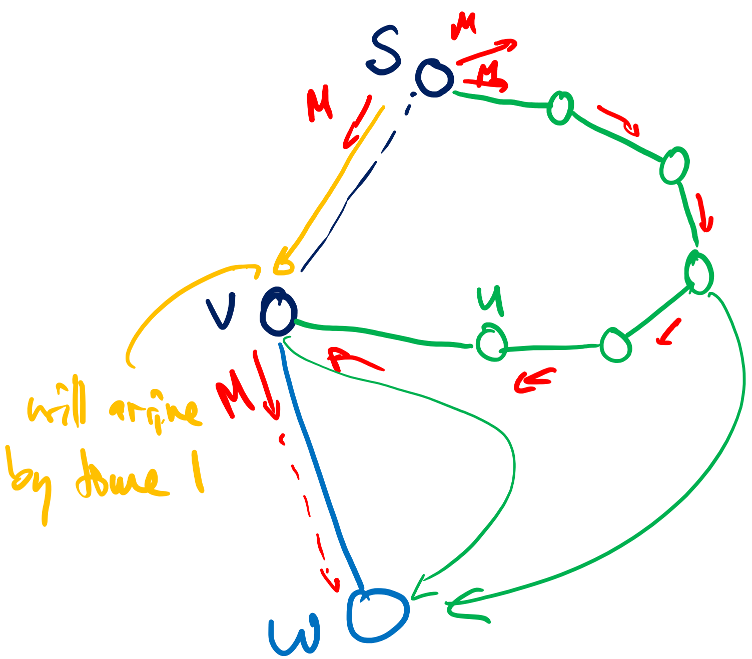
- assign times to send and receive events such that
  - order of all events remains unchanged
  - local computations take 0 time
  - message delays are at most 1 time unit
  - first send event is at time 0
  - time of last event is maximized
- essentially: normalize message delays such that the maximum delay is 1 time unit

Definition Asynchronous Time Complexity:

Total time of a worst-case execution in which local computations take time 0 and all message delays are at most 1 time unit.

Flooding in Asynchronous Systems

Theorem: The time complexity of flooding from a source s in an asynchronous network G is $rad(G, s)$.



by time 1, all neighb. of s know M

⋮

by time r, all nodes at distance ≤ r from s know M.

Message Complexity

Message Complexity: Total number of messages sent (over all nodes)

What is the message complexity of flooding?

Theorem: The message complexity of flooding is $O(|E|)$.

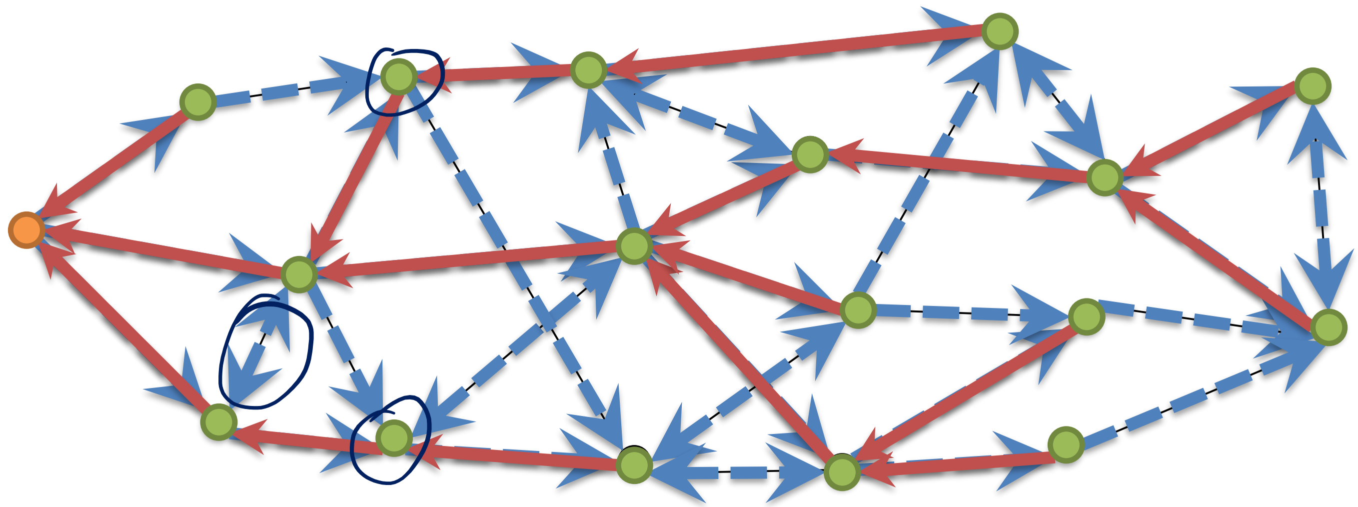
- on graph $G = (V, E)$

Flooding Spanning Tree

- The flooding algorithm can be used to compute a spanning tree of the network.

Idea:

- Source s is the root of the tree
- For all other nodes, neighbor from which M is received first is the parent node.



Flooding Spanning Tree Algorithm

Source node s:

initially do

parent $:= \perp$ // s is the root

send M to all neighbors

Non-source node u:

upon receiving M from some neighbor v

if M has not been received before **then**

parent $:= v$

send M to all neighbors except v

Spanning Tree: Synchronous Systems

- In tree: distance of v to root $=$ round in which v is reached
- In synchronous systems, a node v are reached in round r if and only if $dist_G(s, v) = r$

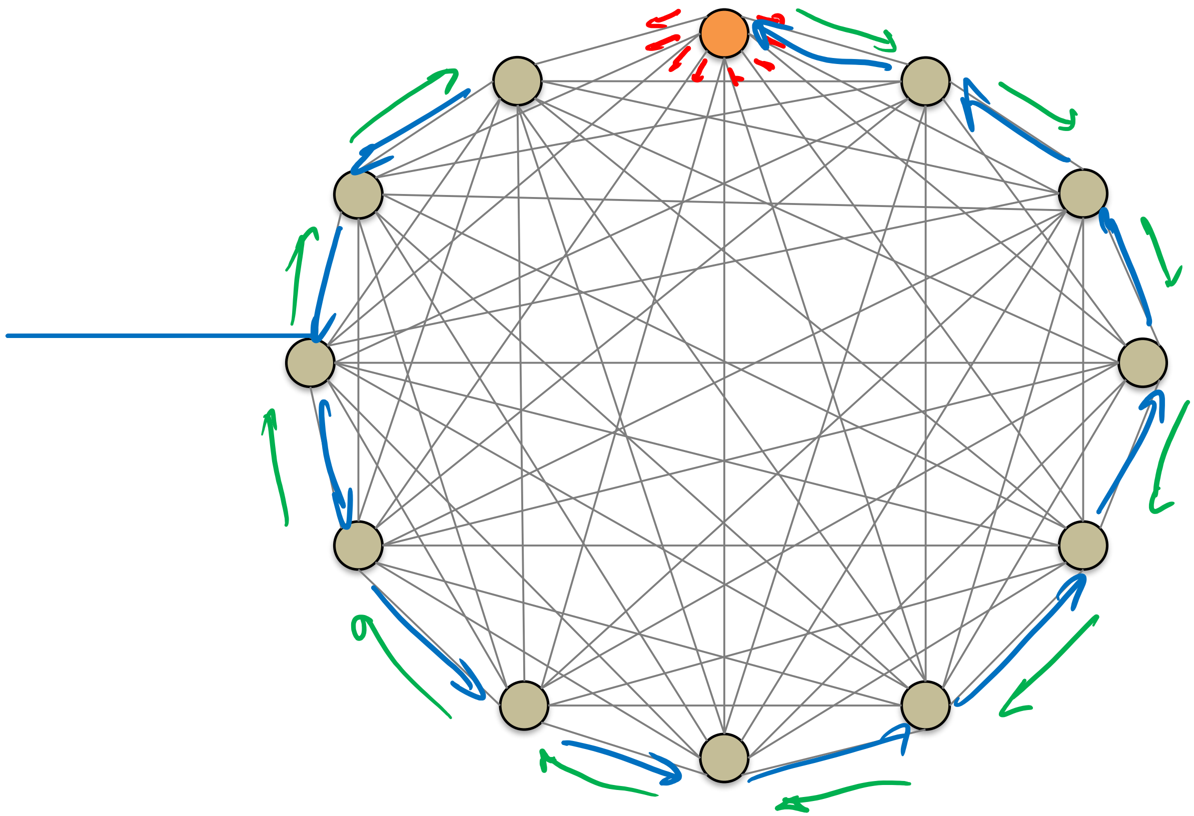
Shortest Path Tree = BFS Tree (BFS = breadth first search)

- tree which preserves graph distances to root node

Theorem: In synchronous systems, the flooding algorithm constructs a BFS tree.

Spanning Tree: Asynchronous Systems

How does the spanning tree look if comm. is asynchronous?

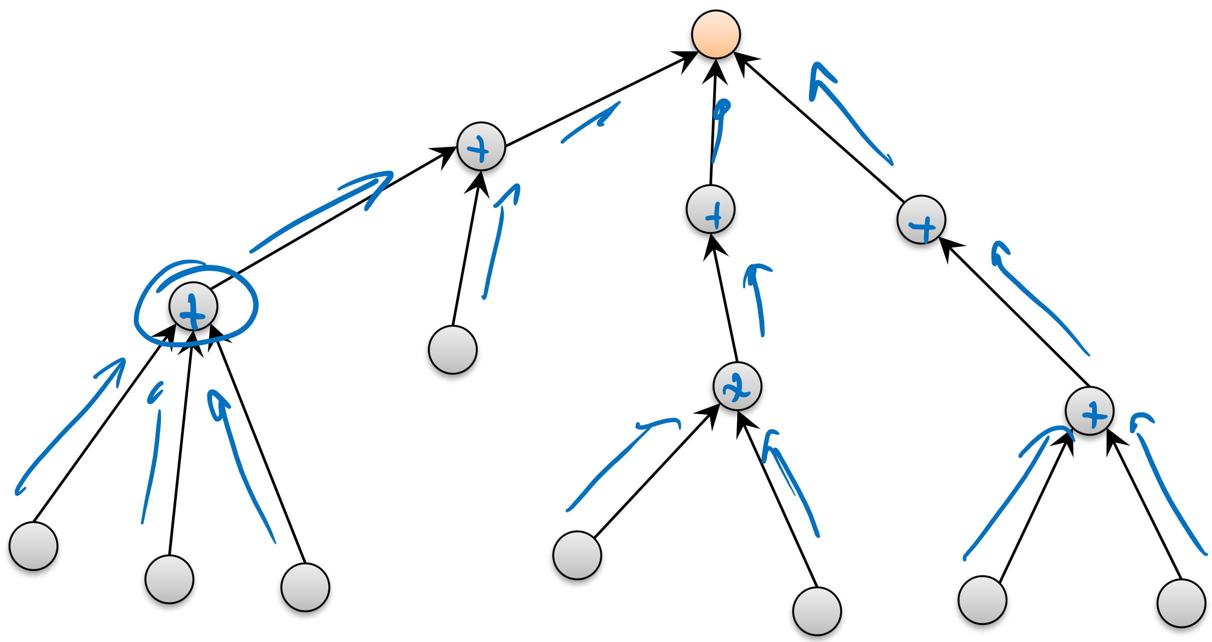


Observation: In asynchronous executions, the depth of the tree can be $n-1$ even if the radius/diameter of the graph is 1.

Convergecast

- "Opposite" of broadcast
- Given a rooted spanning tree, communicate from all nodes to the root
  - starting from the leaves

Example: Compute sum of values in a rooted tree



Convergecast Algorithm

Leaf node v:
initially do
send message to parent
(e.g., send input value)

Inner node u:
upon receiving message from child node v
if u has received messages from all children **then**
send message to parent
(e.g., send sum of all inputs in u's subtree)

Root node r:
upon receiving message from child node v
if r has received messages from all children **then**
convergecast terminates

Convergecast: Analysis & Remarks

Time Complexity:

height of tree

Message Complexity:

#edges of tree = $n-1$

Application of the convergecast algorithm:

- Computing functions, e.g.:
  - min, max, sum, average, median, ...
- Termination detection
  - inform parent as soon as all nodes in subtree have terminated
- ...

Flooding/Echo Algorithm

- If a leader (root), but no spanning tree exists, flooding and convergecast can be used together for computing functions, ...

1. Use flooding to construct a tree
2. Use convergecast (echo) to report back to the root when done

Time Complexity of Flooding + Convergecast (Echo):

Flooding: D — Convergecast: height of tree

$G=(V,E)$

notation

$n = |V|$

$m = |E|$

$D = \mathrm{diam}(G)$

synchronous: $O(D)$

asynchronous: $O(n)$

Constructing Good Trees

- When combining flooding and convergecast, the time complexity is linear in the depth of the constructed tree.
- In synchronous systems, the tree is a BFS tree (shortest path tree), i.e., the depth of the tree is $O\big(diam(G)\big)$
  - optimal time complexity: $O\big(diam(G)\big)$
- In asynchronous systems, the time complexity can be $\Omega(n)$, even if the graph has a very small diameter!
- Convergecast / low diameter spanning trees are important!
- How can we construct a BFS tree in an asynchronous system?

Constructing Shortest Path Tree

Dijkstra

- Grow tree from source s
- At intermediate step t, subtree of all nodes at distance $\leq r_t$ from source node s
- Next step: add node with min. distance to s

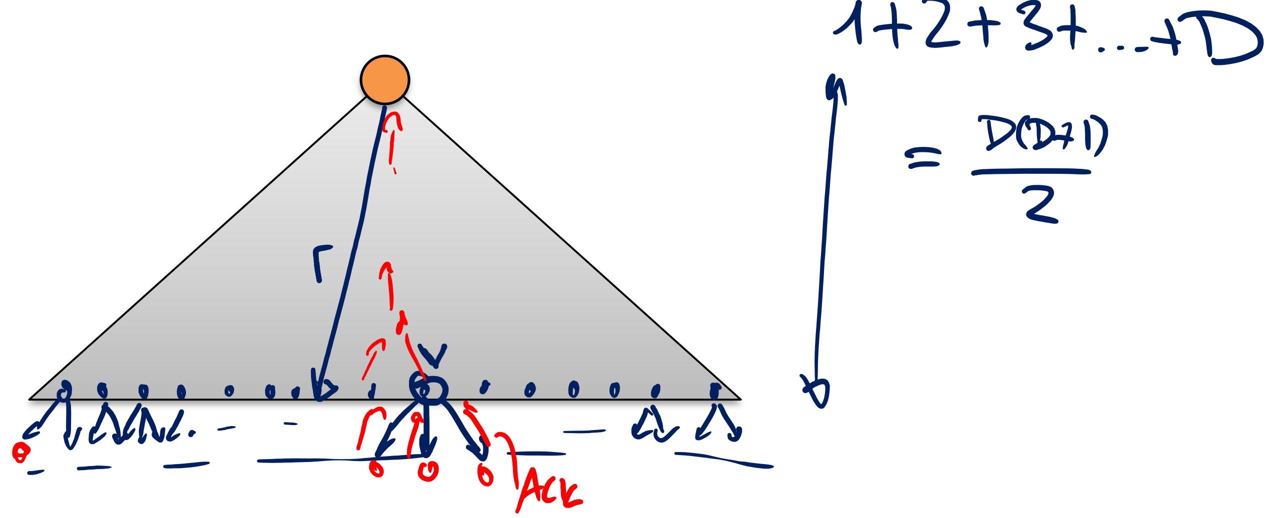
Bellman-Ford

- Each node v keeps a distance estimate d_v to s
  - initially: $d_s = 0,\ d_v = \infty$ (for all $v \neq s$)
- In each step, all nodes update their estimate based on neighbor estimates:

$$d_v = \min\left\{d_v, \min_{u \in N(v)}\{d_u + 1\}\right\}$$

Distributed Dijkstra

- In our case, the graph is unweighted
- We can therefore grow the tree level by level
  - Essentially like in a synchronous execution
- Assume, the tree is constructed up to distance r from s
- How can we add the next level?



Distributed Dijkstra

- Source/root node coordinates the phases

Algorithm for Phase $r + 1$:

1. Root node broadcasts *"start phase* $r + 1$*"* in current tree
2. Leaf nodes (level r nodes) send *"join* $r + 1$*"* to neighbors
3. Node v receiving *"join* $r + 1$*"* from neighbor u:
    1. First such message: u becomes parent of v, v sends *ACK* to u
    2. Otherwise, v sends *NACK* to u
4. After receiving *ACK* or *NACK* from all neighbors, level r nodes report back to root by starting a convergecast
5. When the convergecast terminates at the root, the root can start the next phase

#msg: $O(|E|)$

Distributed Dijkstra: Analysis

Time Complexity:

- time complexity of phase $r+1$? $O(r)$
- overall: $O(D^2)$ $\left[\sum_{r=1}^{D} O(r)\right]$

Message Complexity:

$O(m + D \cdot n)$

join + ACK/NACK

Distributed Bellman-Ford

Basic Idea:

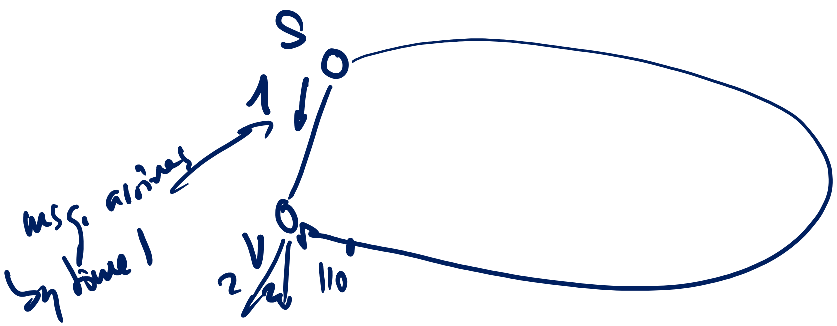
- Each node u stores an integer d_u with the current guess for the distance to the root node s
- Whenever a node u can improve d_u, u informs its neighbors

Algorithm:

1. Initialization: $d_s := 0$, for $v \neq s$: $d_v := \infty$, $\text{parent}_v := \perp$
2. Root s sends "1" to all neigbors
3. For all other nodes u:
   upon receiving message "x" with $x < d_u$ from neighbor v **do**
   set $d_u := x$
   set $\text{parent}_u := v$
   send "$x + 1$" to all neighbors (except v)

Distr. Bellman-Ford: Time Complexity

Theorem: The time complexity of the distributed Bellman-Ford algorithms is $rad(G,s) = O(D)$



Distr. Bellman-Ford: Message Complexity

Theorem: The message complexity of the distributed Bellman-Ford algorithms is $O(m \cdot n)$

msg. per edge & direction

↑

$\leq n-1$

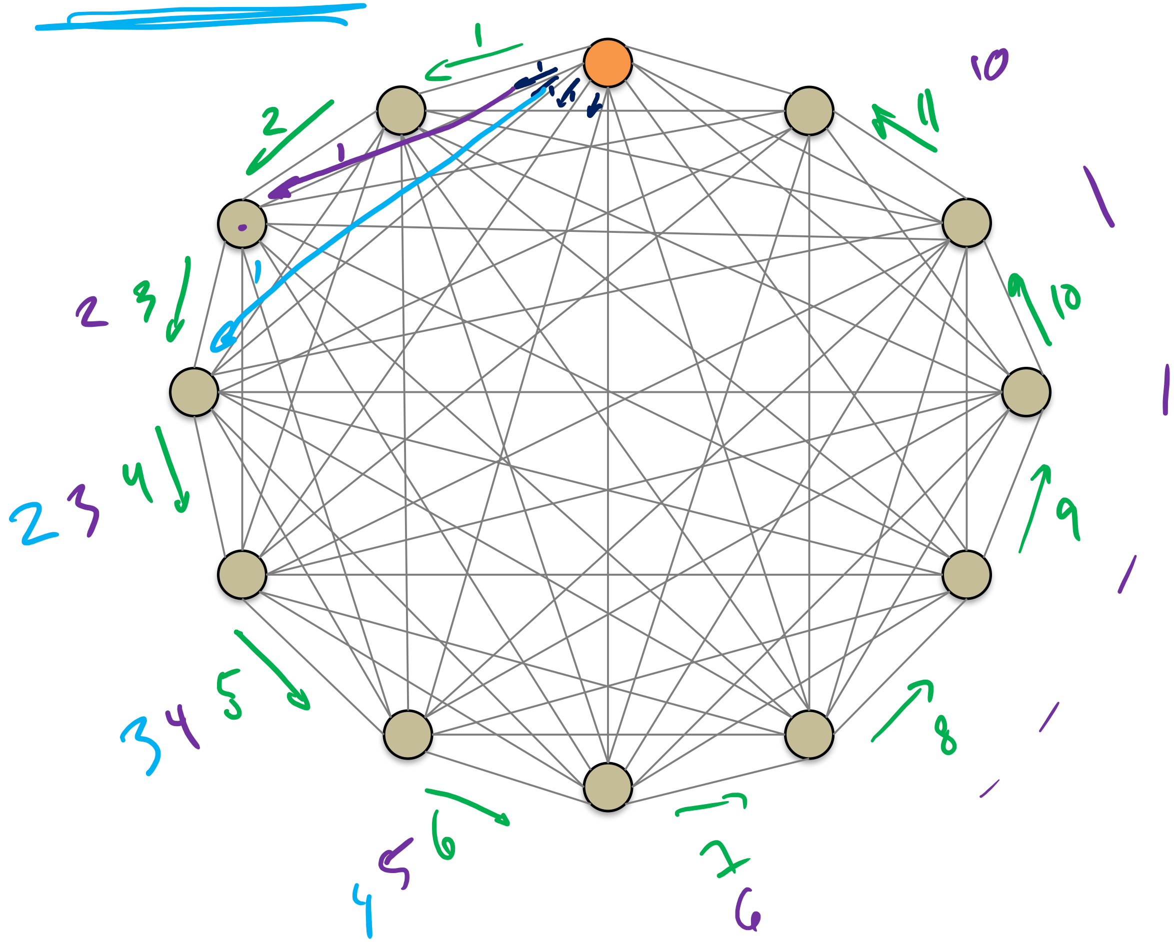
↑

updates per edge

$x \leq n-1$

Distr. Bellman-Ford: Message Complexity

Theorem: The message complexity of the distributed Bellman-Ford algorithms is $O(|E| \cdot |V|)$.



Distributed BFS Tree Construction

Synchronous

- Time: $O\big(diam(G)\big)$, Messages: $O(|E|)$
- both optimal

Asynchronous

- **Distributed Dijkstra:**
  Time: $O(diam(G)^2)$, Messages: $O\big(|E| + |V| \cdot diam(G)\big)$
- **Distributed Bellman-Ford:**
  Time: $O\big(diam(G)\big)$, Messages: $O(|E| \cdot |V|)$
- **Best known trade-off between time and messages:**
  Time: $O(diam(G) \cdot \log^3 |V|)$, Messages: $O(|E| + |V| \cdot \log^3 |V|)$
  - based on **synchronizers**
    (generic way of translating synchronous algorithms into asynch. ones)
  - We will look at synchronizers in a later lecture…